

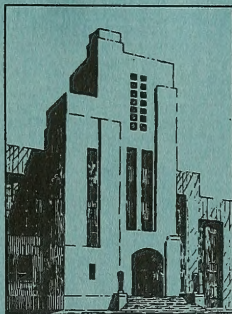
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ANALYSIS OF WAVE RESISTANCE

by

Georg P. Weinblum, D. Eng.



September 1950

Report 710

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Page	Line	From	To
Not.	15	third order	third kind
6	Eq[1]	$C_B = \delta = \frac{1}{2}(C_{BF} + C_{BA}) = (\delta_F + \delta_A)$	$C_B = \delta = \frac{1}{2}(C_{BF} + C_{BA}) = \frac{1}{2}(\delta_F + \delta_A)$
27	19	should be checked	must be checked
	30	Comparing a spheroid	Comparing a wholly submerged spheroid
28	2	formula [18] for	formula ¹⁸ for
35	Fig 14	Delete arrows from 1337 A/2 A/2 and 1370 A/2 A/2	
	Fig 14		
	Title	Insert explanation, - - - before 1337 and — before 1370	
36	Fig 16	Delete ~~$\nabla \phi \eta$~~ from upper left corner	
38	30	data. The ---	data. ¹⁰¹ The ---
39	16	theory for hollow forms with t = 0, the	theory, for hollow forms with t = 0 the
	18,21	K	K (kappa)
40	23,28,31	K	K
	32	F < 0.25	F > 0.25
41	10,15	K	K
50	15	Although the practical	The practical
	16	great, two	great. ¶ Two
81	Eq[3]	y(0,Z) = Z[z]	y(0,z) = Z[z]
84	Eq[32]	K =	K =
88	21	i.e., they ----->0	i.e., they ----->0
90	20	$K_o = \frac{E}{C^2}$	$K_o = \frac{E}{v^2}$
	Eq[8]	Insert after Eq[8]: w = X cos θ + y sin θ	
96	Eq[1]	α - thickness	d - thickness

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NOTATION

A	Area
$A(x)$	Sectional area curve
$A^+(\xi)$	Dimensionless sectional area curve
A as index	Afterbody
B	Beam
C	A constant, coefficients
$E(\eta, H/L, F)$	Resistance function
$F = \frac{v}{\sqrt{gL}}$	Froude's number
F as index	Forebody
G	Resistance function
H	Draft
I	Resistance function
J	Resistance function
J_η	Dimensionless moment of inertia $J_\eta = \int_0^1 \xi^2 \eta d\xi$
K with sub-script	Bessel function of third order
L	Length
M	Resistance function
P	Resistance function
Q	Resistance function
R	Resistance
R_w	Wave resistance
S	Wetted surface
S_η	Dimensionless static moment $S_\eta = \int_0^1 \xi \eta d\xi$
$S(\gamma)$	Resistance function
U	Velocity in the X-direction
X	Axis
$X(\xi)$	Dimensionless equation of water line
$X(x)$	Equation of water line
Y	Axis
Z	Axis
$Z(z)$	Equation of midship section
$Z(\xi)$	Dimensionless equation of midship section
©	Resistance coefficient = $\frac{EHP \times 427.1}{V^3 \Delta^{2/3}}$ V - knots Δ - ts.
a	Longitudinal axis of a body of revolution
a	Coefficients of polynomials
b = B/2	Half-beam, radius
b	Width of a tank

c	Velocity
e	Eccentricity
$e = x_0/L$	Ratio of distance of CB from midsection
f	Depth of immersion
g	Gravity acceleration
h	Finite depth of water
k	Integer, coefficient
$k(x)$	Equation of centerplane contour
$l = L/2$	Half-length
m	Integer, exponent
m	Intensity of doublet distribution
m	Variable of integration
n	Integer, especially exponent
q	Intensity of source-sink distribution (output)
r	Resistance coefficient
$t = - \left \frac{\partial \eta}{\partial \xi} \right _{\xi = \pm 1}$	Taylor's tangent value
v	Speed
x	Coordinate
x_0	Coordinate of a centroid
y	Coordinate
$y = y(x, z)$	Equation of surface
$y = y(x)$	Equation of water line
$y = y(z)$	Equation of section
z	Coordinate
$\alpha = C_w$	Water line coefficient
$\beta = C_m$	Midship section coefficient
γ	Variable of integration
$\delta = C_B$	Block coefficient
ϑ	Variable
$\zeta = \frac{z}{H}$	Coordinate
$\eta = \frac{y}{\frac{B}{2}}$	Coordinate
$\xi = \frac{x}{\frac{L}{2}}$	Coordinate
	} Dimensionless
$\eta = \eta(\xi, \zeta)$	
$\eta = \eta(\xi)$	
$\eta = \eta(\zeta)$	Dimensionless equation of section

η_s	Symmetrical	} parts of η
η_a	Asymmetrical	
λ	Wave length	
$\mu = \frac{m}{4\pi}$	Moment of doublet	
$\sigma = \frac{q}{4\pi}$	Strength of source	
ρ	Density	
ϕ	Prismatic coefficient	
$\psi = \textcircled{M} = \frac{L}{\nabla^{1/3}}$	Ratio of slenderness	
$\gamma_0 = 1/2 R^2$		
Δ	Weight displacement	
∇	Volume displacement	
ϕ	Resistance function	

ANALYSIS OF WAVE RESISTANCE

by

Georg P. Weinblum, D.Eng.

ABSTRACT

The purpose of the report is to summarize the contributions made by hydrodynamics to solve problems of wave resistance of ships moving with a constant speed of advance. A necessary condition for dealing with the problem is the introduction of mathematically defined ship lines. The difficulties in determining the wave resistance experimentally and theoretically are discussed. Endeavors are made to work out separately the influence of proportions and the dimensionless shape on wave resistance; as a most important result, from the viewpoint of shipbuilding practice, the proof is obtained that under certain conditions small continuous changes of longitudinal-displacement distribution involve large changes in wave resistance. The influence of the vertical-displacement distribution and of special features such as the hull are investigated. Consideration is given to the propeller action and to resistance in shallow water and in canals (basins). Wholly submerged bodies of revolution are dealt with.

Appendices 1 and 2 contain a list of formulas used and Appendix 3 some remarks on viscous pressure resistance.

INTRODUCTION

The problem of wave resistance is perhaps the most interesting if not the most important problem in theoretical naval architecture. Our great ship-research laboratories were founded and originally developed for the purpose of investigating essentially this part of ship resistance, and a tremendous amount of experimental data on models have been collected. Nevertheless, the long nourished aim to represent the resistance of a ship as a function of her form has not been solved in a general manner by such experimental methods, although systematic model tests have been by far the most important means of improving the shape of hulls up to the present time.

Concurrently theoreticians, beginning with Lord Kelvin,^{24*} have tried to solve the problem of the wave resistance of floating bodies by classical hydrodynamic methods. Starting with the simplest abstractions, which are suitable to describe the phenomena only in the most general and tentative way, they developed theories which are claimed to be useful for practical design work. No clear conclusion has yet been reached as to how far this claim is justified. Some scientists who have made important contributions to the subject hold the opinion that the existing theories give only a good qualitative picture of the actual facts, while other authors claim that close quantitative agreement can be reached.

In 1925, Professor Havelock delivering a lecture on "Some Aspects of the Theory of Ship Waves and Resistance"⁷⁷ quoted a paragraph from Kelvin's famous paper on ship waves ("I made it a condition to the Council that no practical results were to be expected from it. I explained that I could not say one word to enlighten you on practical subjects.....") and appropriated this statement to his own work. Havelock could well afford to underestimate the practical importance of his work because it is in fact his paper, together with a contemporary article by C. Wigley,⁸² that marks the beginning of the application of theory to practical discussions on wave resistance. The lecture mentioned and Wigley's reviews on the subject are recommended for study to all students of the subject.^{84, 85}

The present paper is to be judged from the viewpoint of how far it is able "to enlighten on practical subjects..;" its purpose is to show to what extent theory has succeeded in furnishing valuable practical results and how the scope of its applications can be extended. It is not a treatise on hydrodynamics dealing with problems of free-surface effect; rather an attempt to reconsider a basic phase of theoretical naval architecture in the light of hydrodynamics. It bears by necessity a somewhat programmatic character because the tedious computations connected with the evaluation of the theoretical work have been initiated but not yet completed.

There is common agreement that theory has furnished a valuable description of general phenomena; it is less well known that it also has given us the proof of considerable practical value of how sensitive wave resistance can be to changes, even small changes, in ship form.

Experienced experimenters are often somewhat bewildered by the fact that the wave resistance may vary appreciably for different but reasonable types of lines, although all the form parameters generally considered as decisive are identical. From a theoretical viewpoint this appears to be quite

*References are listed on page 97.

natural, since the wave resistance depends to a first approximation upon a complicated function of the surface slope in the longitudinal direction, i.e., on derivatives. On the other hand, the most commonly used coefficients are integrals, which even when kept constant still admit of very wide variations of the slopes. We realize now why the solution of the basic problem of the model tanks mentioned above—to establish the resistance as a function of the form—remains almost hopeless as long as the ship surfaces (or at least their most important features) are not defined in a rigorous way by mathematical expressions. Hence, our first task must be to find equations for the ship surface, continuing the work of D.W. Taylor.⁴⁵ This phase of Taylor's research has been more or less neglected by later investigators.

The well known resistance phenomena are briefly reviewed herein, and the formal procedures are set out by which hydrodynamics leads to a computation of wave resistance.*

The main purpose of the report is to establish simple functional relations between resistance and form of "normal" ships, based on Michell's theory and on a comparison of calculated and measured resistance values. This includes a discussion of the effects of viscosity and of some proposals intended to improve the theoretical procedure.

It was further thought useful to mention the interaction between ship and propeller and to study the effects of restricted water, especially with respect to the influence of the finite dimensions of model basins, on model wave resistance.

Finally a synopsis of current knowledge regarding the wave resistance of wholly submerged bodies is included.

Wave resistance phenomena in a seaway have not been treated to avoid making the report too long, although theory has recently made valuable contributions to this thrilling subject. For the same reason the problem of nonuniform motion has been omitted.

When emphasizing the importance of theoretical work, it is necessary to state that the theory is based on many abstractions and that the combination of theoretical and experimental work is a necessary condition for reaching useful practical results.

*The subject of the report is the analysis of wave resistance. Therefore for brevity "resistance" is often used instead of "wave resistance;" when other "kinds" of resistance are discussed the fact will be clearly stated.

1. THE REPRESENTATION OF SHIP FORMS

1.1. THE GEOMETRY OF SHIPS

Any treatise on theoretical naval architecture should begin with (or at least include) a chapter on the representation of ship forms. This could be called "The Geometry of Ships," although this terminology has sometimes been used for discussions of problems of statical stability.

1.1.1. Graphical Method

The generally used graphical method of representing ship surfaces by "fairing" the lines is efficient from some viewpoints, as it leads within a reasonable time to solutions which comply with the necessary conditions of buoyancy, stability, etc. The resulting surfaces (lines) have a high degree of smoothness within the practical degree of accuracy required when "spline curves" (battens) are used. A curve is called "smooth" when the first derivative is continuous (the curve itself has no corners); we define the "order of smoothness" of a curve as the order of the highest derivative which is continuous. A curve in which there is a continuous radius of curvature (a continuous second derivative) is smooth to the second order. Spline curves drawn in the proper way should generally be smooth to a still higher order. This can be easily understood since the curvature of the elastic line of a batten is proportional to the bending moment; the graph of the bending moment remains continuous, even if we apply to the batten horizontal concentrated loads by weights—a procedure contrary to the idea of fairing under normal conditions. On the other hand, discontinuities in the curvature of ship lines are admitted, for instance, when using a combination of a straight line and a circular arc for sections.

These points will be considered later;* at the present it is sufficient to state that a definite order of smoothness may be a necessary or desirable condition for a ship line, but it is not a sufficient condition as is well known from experiments and will be proved by theory. It is the purpose of resistance research to develop criteria for good ship lines—lines determined by minimum resistance qualities. Because of the lack of rigorous results, however, earlier practice—guided by experience and other considerations, some of arbitrary character—has introduced a working concept of "fairness of lines," with which hulls should comply.

*See Appendix 3, where it is shown that modern hydrodynamics supports the wisdom of artisanship in shipbuilding.

It is not easy to give a reasonable definition of the real meaning of "fairness." As applied to a curve some features characterizing this property are: (a) Conditions of monotonic increase or decrease, (b) avoidance of flat parts, (c) lack of abrupt change in the curve and its first and second derivative, and (d) the possession of a finite number of points of inflection.

The basis of the idea of fairness is the fact that a trained human eye is very sensitive to any peculiarities of curves. It will be proved later that fairness has not given us reliable indications as to the resistance qualities of forms, although the concept can be useful from resistance and structural viewpoints. The well known resistance qualities of bulbs and "swan-necks" furnish a good illustration of how far the idea of fairness must be modified in the light of present knowledge.

The disadvantages of the graphical method of ship design are: (a) The necessity of laying off to a large scale, and (b) the unsuitability of graphical representations for establishing general laws for wave resistance, behavior in a seaway, and so on.

1.1.2. Description by Coefficients and Lines

The "Explanatory Notes for Project Number 2"⁵² issued by The Society of Naval Architects and Marine Engineers contain a consistent system of form parameters and lines which are assumed to be known.

Use is made of dimensionless representation: Dimension factors are separated from the pure form or shape defined as the ratio of hull ordinates y to half-beam $B/2$, etc. The dimensions can be expressed by the absolute length L and two ratios as $L:B, B:H$; the shape can be approximately described by suitable integral or differential properties (coefficients of fineness, tangents, curvatures). Thus two sets of important parameters are obtained which, within a certain range, can be treated independently, as will be demonstrated. In the light of this tendency there does not seem to be any advantage in the use of such expressions as $\frac{L}{\nabla^{1/3}}, \frac{\nabla}{L^3}$, where the pure form constants and the parameters derived from dimensions are mixed, except for a first orientation; for practical work the use of ratios, like $L/B, B/H$, and $C_B = \delta$ (which are all known) is preferable.* Our purpose is to approximate the ship form by as many characteristic parameters as possible, not to merge several known parameters into one.

*As a matter of courtesy, symbols like C_B, C_p , proposed by the Sub-committee of the SNAME, are mentioned here beside the well established Greek letters which are used throughout the report. For actual work Greek letters have a definite advantage over the C coefficients by reducing the prohibitive number of subscripts.

It is definitely advantageous to use fore and afterbody coefficients like

$$C_B = \delta = \frac{1}{2} (C_{BF} + C_{BA}) = (\delta_F + \delta_A) \quad [1]$$

The differences $(C_{BF} - C_{BA})$, $(C_{pF} - C_{pA})$ can be connected to a good approximation with the ratio $\frac{x_0}{L}$, where x_0 denotes the distance of the L.C.B. from the midship section. (It is an offense against the spirit of approximate calculus to refer the longitudinal position of the CB to the fore or aft perpendicular.) The normal ship form consists of a main part symmetrical with respect to the midship section roughly characterized by C_B , C_p and an asymmetrical part described by

$$\frac{C_{BF} - C_{BA}}{C_B} = \frac{\delta_F - \delta_A}{\delta} = k \frac{x_0}{L} \quad [2]$$

where k is a factor depending on the fineness and the form of the sectional-area curve. Values of coefficients depending on the longitudinal distribution of volume, etc., are useless if the reference lengths (L_{PWL} , L_{PP}) are not clearly stated. Where $L_{PWL} \neq L_{PP}$, both values like $C_{BW}(\delta_W)$ and $C_{BP}(\delta_P)$ should be given.

The basic form coefficients have been developed with respect to problems of buoyancy and stability; it is a fortunate coincidence that they are characteristic for resistance problems too. Parameters introduced from hydrodynamic considerations are:

$$a. \quad t = \frac{L}{B} \tan \gamma = \frac{L}{B} \left| \frac{\partial y}{\partial x} \right|_{x = \frac{L}{2}} \quad (\text{Taylor's tangent value}) \quad [3]$$

b. Length of parallel body.

c. Position of the point of inflection.

d. Curvature at the midship section.

e. Position of the centroid of the forward and aft parts of a waterline (sectional-area curve).

f. Bulb-area ratio f defined by Taylor as the ratio of intercept of the sectional-area curve at the bow to the maximum ordinate.

Items c to e have not been much used. The list may be increased by using:

g. Moments of inertia, or

h. Higher moments with respect to suitable axes.

However, graphical procedures become cumbersome when all these coefficients have to be considered, so that it is preferable to introduce mathematically defined ship lines.

A great success has been achieved in naval architecture by the use of integral curves of which the sectional-area curve is perhaps the most important.

1.1.3. Equations of Ship Surfaces

Quite a number of attempts have been made to define ship lines by mathematical equations, with different purposes in view. There is no need to dwell on Chapman's parabolas, which can be found in every handbook. Obviously they are almost impractical, as they contain only one parameter, i.e., their shape is fixed by the area coefficient alone. One should therefore eliminate them as a method for design, notwithstanding the fact that they have formed the starting point for all further work on the exact representation of ship form.

According to D.W. Taylor's own statement,⁴⁵ he developed "Mathematical formulae not with the idea that they give lines of least resistance but simply to obtain lines possessing desired shape." This statement is important; contrary to some attempts to ascribe magic properties to certain analytically defined curves like trochoids, sine curves, etc., the principle of systemization is put forward as the decisive argument for their adoption. Formulas are given for waterlines and sections separately, no parallel body having been apparently envisaged, although there is no difficulty in inserting this feature. Fore and afterbodies are treated separately; the origin is situated at the bow or stern, respectively. The family of curves intended to represent waterlines and sectional-area curves is given by a fifth-degree polynomial with three arbitrary parameters; the discussion of the properties of this family is performed in classical style and is perhaps the most refined contribution to our problem. The lines are perfectly suitable for practical use.

Unfortunately, Taylor's work is not very suitable for theoretical investigations since the equations of waterlines and sections are not linked together into equations of surfaces and fore and afterbody are treated separately. Nevertheless, it is a great loss to the profession that his basic publication⁴⁵ has remained almost unknown; otherwise model research would have obtained a more systematic character, as earlier stated.

A system of ship-hull equations has been developed for broader purposes^{103, 104} in which expressions were sought which would:

- a. Be suitable for fixing and systematizing ship forms.
- b. Give a basis for investigating systematically wave-resistance qualities of ships and eventually lead to forms of least (low) resistance.

c. Form a more general foundation for research on other problems in naval architecture such as motions in a seaway, stability, etc.

d. Reduce the work in the mold loft. (The author has made some contributions to this idea without being able to claim much practical success.)

The formal development is briefly reviewed in Appendix 1. Although the analytical approximation of a given hull may be a tedious problem, for some scientific purposes quite simple expressions prove to be valuable.

We locate the origin at the midship section, and the planes of reference are given by the vertical-center plane (X,Z), the load-waterline plane (X,Y) and the midship-section plane (Y,Z); see Figure 1. In dealing with resistance problems to a first approximation, we confine ourselves to the under-water part of the hull.

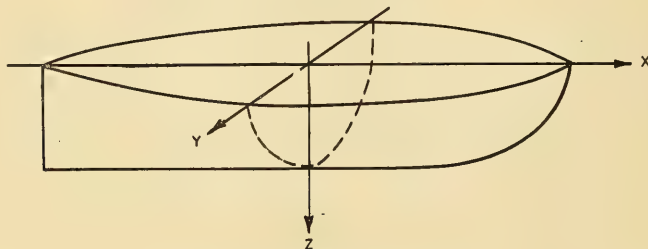


Figure 1 - Axes of Reference

We denote as an "elementary" hull a form defined by a rectangular longitudinal contour and an equation of the type

$$Y = \frac{\beta}{2} X(x) Z(z) \quad [4]$$

where $X(x)$ and $Z(z)$ are the dimensionless equations of the load waterline and midship section. It can be easily seen that for such hulls:

$$a. \quad C_B = \delta = C_W \times C_M = \alpha \beta \quad [5]$$

- b. The sectional-area curve is affine to the load waterline (their equations differ only by a constant factor)

$$C_P = \phi = C_W = \alpha \quad [6]$$

- c. The sections are affine to the midship section and the waterlines to the load waterline.

It is astonishing that such an elementary equation leads to quite reasonable ship forms as long as the $C_M = \beta$ relationship is low; it is appropriate for investigating the influence on wave resistance of the longitudinal

and, within certain limits, of the vertical distribution of displacement. By introducing additional functions we are able to represent changes in different sections (for instance, inclination to the vertical, variation of fineness, etc.) and waterlines.*

Analytical representation of ship forms, developed for the purpose of eliminating the mold loft, meets with two essential difficulties:

1. The longitudinal profile is generally not a rectangle but a curve, often with corners and discontinuities. This complication probably can be overcome by a method already proposed.¹⁰³
2. The representation of full sections. Here the use of high-degree parabolas leads to regions with high curvature, which are detrimental from a hydrodynamic as well as a practical viewpoint. Some improvements can be made, but as long as this difficulty is not overcome, it is not suggested that the formulas be applied for construction. However, readers interested in the problem may refer to a paper by Childsky,⁸ where a criticism of the present methods and indications of some further development may be found.

An important question arises: Should the usual method of ship design, based on the sectional-area curve $A(x)$, design waterline $X(x)$, longitudinal contour $C(z)$ and midship section $Z(z)$, be altered when using analytical expressions? Within the present range of application there seems to be no need for a radical change. Of course, the polynomial $A(x)$ must have a sufficiently high degree to comply with all functions of x representing the ship's surface, and a further difficulty arises when the longitudinal contour departs from a rectangle, because then the equation of the hull cannot be easily expressed by a polynomial.

The use of algebraically defined surfaces enables us to fit ship forms in a rigorous manner, and especially to describe in a definite way changes and variations in the forms. In applying the equations to problems of resistance, motion in a seaway, etc., we expect to deduce results capable of generalization and to establish parameters which are characteristic for the problems involved. An interesting, if not very important, application of mathematical ship lines is the development of reliable approximate formulas for ship design (position of centroids, moments of inertia) as already indicated¹⁰⁴ and extended by Sparks.³⁶

*Examples of waterlines and sectional-area curves are given in Figures 15, 16, and 17 on pages 35, 36, and 41.

2. RESISTANCE

When analyzing the wave resistance, some remarks on ship resistance in general are necessary. We ignore the air resistance, which may be quite an important item but is outside the scope of these considerations.

Froude developed his well known method of using model experiments by starting from the idea that it is comparatively easy to calculate the frictional drag and impossible to compute the wave resistance. He tacitly assumed that the different "components" of resistance can be superposed with reasonable accuracy, i.e., that their mutual influence is relatively the same for the model and the ship. The "eddy" or "separation" resistance, although obviously due to viscosity, is merged into the residual resistance and its coefficient considered to be independent of scale effect; it was assumed to be small for good ship forms.

These assumptions have led to a practical technique used all over the world. From the viewpoint of hydrodynamics, however, matters are somewhat different. Here the frictional resistance due to viscosity effects is in principle a more complex problem (involving the solution of Navier-Stokes equations) than the wave making force which arises in an ideal fluid. The generation of surface waves in a viscous medium is a problem which has not yet been investigated by theory, so that the hypothesis of the mutual independence of the resistance components from scale effect (model and ship) cannot be quantitatively discussed in the light of rational mechanics.

The "eddy" resistance due to viscosity is a function of some appropriate Reynolds number. From the reasoning given in Appendix 3, a serious scale effect can be expected when model data are converted to the ship, although it is assumed that the character of separation does not change when full turbulence has been reached over the model surface. In aerodynamics, contrary to Froude's procedure, frictional and "eddy" resistance are treated together as viscous drag.

These undetailed remarks indicate how difficult it is to analyze rigorously the wave resistance of the ship using model experiments. For most types of ships wave resistance is only a small part of the total resistance and as the amount of viscous drag is known only to a relatively low order of accuracy, the wave-resistance values obtained by subtraction are liable to be quite erroneous.

To put things on a more rigorous foundation we define the following kinds of resistance:

a. Frictional resistance is the tangential resistance due to viscous forces. It differs in principle from the resistance of the equivalent plate having the same wetted surface, this difference being denoted by frictional

form resistance. So long as no separation occurs, a lower limit for this frictional form resistance can be estimated from Millikan's formula for bodies of revolution,⁵⁹ which gives an increase of the order of 4 to 6 percent over the corresponding plate values. A further effect is the change in the wetted surface due to the changed attitude of the model and to the wave flow. The latter is important for high Froude numbers only;⁸⁴ it becomes decisive for vessels with large hydrodynamic lift.

b. Viscous normal or pressure resistance: This resistance is caused by the pressure changes due to viscosity, especially in the afterbody. It includes separation effects and loss of pressure due to finite boundary thickness. This definition is broader than "separation" or "eddy" resistance; the addition "viscous" becomes necessary to avoid confusion with wave resistance which can also be derived from pressure measurements. Frictional and normal resistances—a and b above—can be separated by pressure measurements when wave making is absent or negligible.

Throughout this report the denotation "eddy resistance" will be used in the same sense as viscous normal resistance. Sometimes the sum of frictional-form resistance plus viscous-pressure resistance is referred to as viscous-form resistance; the adjective viscous should not be dropped.⁴⁸

c. Wave resistance: Items a and b give the total viscous drag. Wave resistance can be computed as the difference between the total normal and the viscous normal resistance, or between the total resistance and the total viscous drag. Surface models and wholly submerged double models have been used for this purpose under the assumption that for the same Reynolds number the wave flow does not influence a and b.^{2, 14, 98} One can dispense with the double models if pressure measurements on large simple models admitting of high Reynolds numbers are made at low Froude numbers giving a negligible wave pattern. A more direct approach would be to calculate the resistance from actual wave patterns (stereo-photogrammetric pictures) which, however, never has been successfully tried. Most of these methods are suitable for research work only.

d. Spray or jet resistance: This becomes important for very-high-speed length ratios, such as are found in gliders and seaplanes. It must be listed here as a surface effect sui generis which is not described by the theory of surface waves; in fact its generation does not depend on gravity. In normal vessels, sprays can be prevented by avoiding blunt surfaces normal to the direction of advance.

e. Thrust deduction or suction force due to the propeller: Generally this force is not considered when dealing with the resistance, since it depends on the interaction between propeller and hull. However, it seems to be appropriate to mention it, as it influences decisively the form of the afterbody, especially for single-screw ships, by requiring the creation of a wake axially symmetrical with respect to the propeller shaft.

f. Resistance in a seaway: The increase of resistance in a seaway is mainly due to the oscillatory motions (pitching and heaving),^{8,2} and, judging from results for a fixed ship,^{7,4} the reflection of waves is responsible for only a small part of the resistance increase. The damping of oscillations is primarily caused by wave making.

Older experiments led to the belief that ships with good resistance qualities in calm water keep their superiority in regular waves. There are, however, some indications that ships with a steep rise of the calm-water resistance curve near the service speed are liable to high resistance in rough water. This remark applies especially to hulls with high prismatic coefficients.

Summarizing the state of knowledge on some problems of ship resistance—so far as it is needed for analyzing wave resistance—we may say:

a. The residual resistance derived by Froude's method is highly artificial, including as it does items which do not follow Froude's law of similitude. For small Froude numbers the "residual" resistance may not yield even an approximate estimate of wave resistance.

b. No reliable results are known for the viscous pressure resistance; even contradictory statements are to be found as to its range.^{2, 14} From measurements on bodies of revolution fully submerged, we can guess that the total viscous drag of models up to moderate prismatics corresponds closely to the frictional drag including the frictional form effect.*

c. An accurate estimate of the total resistance of slow ships depends more on the reliable computation of frictional resistance including the roughness and form effect than on accurate values of the residual resistance derived from model results. Thus the routine towing experiments would not seem to be indispensable for this most important class of ships as long as the computation of absolute value of resistance is the main problem; nevertheless,

*Special investigations should be made as to the dependence of the eddy resistance (viscous pressure resistance) upon the form of the ship, especially the afterbody (see Appendix 3). Various rules have been made for the avoidance of appreciable eddymaking;⁴ unfortunately they are more a matter of opinion than of real knowledge. Some deductions are derived from inadequate experiments, for instance as to bodies of revolution where the influence of the free surface was not eliminated, etc.

tests are valuable from the viewpoint of obtaining reliable comparative results for various forms. Accurate absolute values of wave resistance cannot be obtained by Froude's method for low speeds, but we can hope to obtain (by using great care) consistent differences in wave resistance corresponding to definite changes in form; much is achieved when we can rely on the sign of the result. Unfortunately, the same doubts as to the absolute values apply to the calculated wave resistance in this range. Our main task will be, therefore, to establish whether changes in the calculated wave resistance do correspond to the experimental values with respect to the general trend and range, not so much with respect to the absolute values, although the agreement in the latter sometimes is also very satisfactory.

3. GENERATION OF BODIES BY SINGULARITIES AS BASIS FOR CALCULATING WAVE RESISTANCE

The representation of ship forms by algebraic expressions enables us to calculate the wave resistance comparatively easily for the limiting case of a slender body, the so-called "Michell's ship." The assumption is made that the slope of the hull to the vertical center plane expressed by $\frac{\partial y}{\partial x}, \frac{\partial y}{\partial z}$ is everywhere small.

While the condition for the longitudinal slope $\frac{\partial y}{\partial x}$ is not unreasonable for most normal ship forms, the condition $\frac{\partial y}{\partial z}$ is very seldom satisfied. Let us investigate, at least superficially, the assumptions on which the theory is based.

Lord Kelvin²⁴ substituted for the ship a pressure concentrated at a point and applied to the surface which he designated by "forcive." He succeeded in explaining the main features of the wave pattern created by a ship. His work was widely extended by E. Hogner¹⁸ who gave a general formula for the wave resistance of pressure systems, which enables us to calculate the resistance of bodies with negligible draft, especially gliding craft, when the pressure distribution is known.

For normal ship forms the approach by the usual singularities—sources, sinks and doublets—has proved to be more appropriate. The generation of bodies by sources and sinks (doublets), proposed by Rankine,³⁵ should always form a basic study in theoretical naval architecture. Before discussing bodies floating on the surface, we will investigate wholly submerged bodies in an unbounded fluid (the free surface is very far away from the body).

It is well known that under appropriate conditions, closed slender bodies of revolution are generated by sources and sinks (doublets) located

along a straight line in a uniform stream. Weinig's important approximation holds: That the intensity of the doublet distribution is roughly proportional to the sectional-area curve of the body.⁴⁹

We now distribute surface singularities over an elongated region of a vertical plane (Figure 2). Again in a horizontal uniform stream a closed surface is created, if the integral of the strength of the source and sink system, taken over the given region of the plane, is zero. One can prove that, within certain limitations, the surface ordinate y_0 of the resulting body at a point x_0, z_0 , is roughly proportional to the doublet strength at the point, and that the sectional area curve $A(x)$ is approximately proportional to the

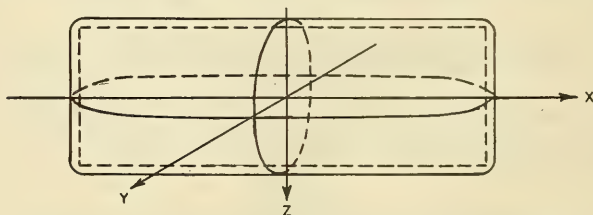


Figure 2 - Generation of Bodies by Singularities

integral of the doublet intensity taken over z . Under the same conditions the strength of the sources and sinks is proportional to the slope of the surface in the horizontal direction.

Let the distribution over the plane be symmetrical with respect to the x -axis, then the resulting body is also symmetrical with respect to the xy plane. Within reasonable assumptions as to its character we can assert that the maximum beam $2b$ is smaller than the height $2H$. This is easily understood in the limiting case of a body of revolution when the singularities are concentrated on the axis; by definition $2b = 2H$. By displacing an element of the distribution away from the x -axis in the direction of the z -axis we obviously increase the vertical dimension of the resulting body.

To generate bodies characterized by $2b > 2H$ one must obviously distribute singularities in a horizontal (x, y) plane. Some quantitative relations may be estimated from the known results for a general ellipsoid; Figure 3. Assuming $a > c > b$, i.e., the vertical axis greater than the horizontal axis in the y direction, it has been proved that the ellipsoid is generated by a doublet system $M(x, z)$, which is distributed over the focal ellipse in the xz plane

$$\frac{x^2}{a^2 - b^2} + \frac{z^2}{c^2 - b^2} = 1$$

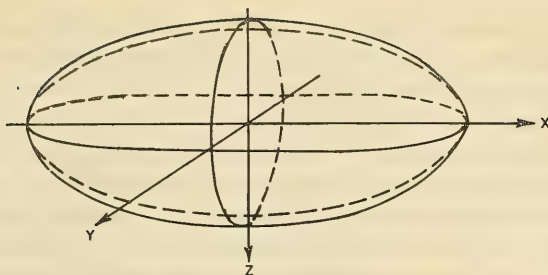


Figure 3 - Ellipsoid

with the moment

$$M(x,z) = C \sqrt{1 - \frac{x^2}{a^2 - b^2} - \frac{z^2}{c^2 - b^2}} \quad [8]$$

where C is constant. The ellipsoid is a simple example of a "double" model.

This somewhat lengthy reasoning on submerged bodies has been made with the purpose of obtaining an approximation for surface ships. It is assumed that the underwater part of the ship is one half of the "doubled" submerged body and is generated by half the corresponding singularity distribution, an assumption which is rigorous only when the wave making can be neglected. The approximate relations between the body form and the image system are also applicable to the ship under this condition.

A closer approximation can be obtained by distributing singularities over the surface. As no useful practical results have been obtained in this way we shall not pursue this method at present.

"Michell's ship" corresponds to an image system distributed over a vertical plane. Actually ships with $b = \frac{B}{2} > H$ should be approximated in another way, which has been initiated by Havelock⁷⁹ and used by Lunde.³¹ Nevertheless, our reasoning will be based on Michell's formula³² which is valid under the following conditions:

- a. Ideal fluid.
- b. Small slopes of the surface.
- c. Small wave slopes.
- d. No change of model attitude.

In using this reasoning, we can expect good agreement between theory and experiments for slender bodies only and depend upon experimental checks and corrections for ships of normal proportions.

Originally Michell developed his resistance formula by computing the pressure exerted on the body.³² Other methods are based on the computation of the energy of the wave system caused by the motion of a body and on the dissipation of energy calculated by means of an artificially introduced vanishing viscosity term introduced by Havelock.^{26, 69, 70} A fourth approach used by Havelock⁷² is the so-called method of singularities, which we shall review briefly because of its advantages for calculating forces when the image systems involved are known.

The method of singularities may be simply explained as the law of attraction applied to sources and sinks: Two point sources (or sinks) with the output $Q_1 Q_2$ ($-Q_1$, $-Q_2$), attract each other with a force

$$K = \frac{Q_1 Q_2}{4\pi r^2} \quad [9]$$

where r is the distance between the two singularities, while a source and a sink experience a repulsion of the same absolute value.³³ It is quite astonishing that no broader use has been made of this formula in hydrodynamics, which when applied to electricity is familiar to any student.*

The formula [9] can be rewritten and generalized to give the force experienced by a source Q due to the velocity v of the stream at the location of the source

$$K = -\rho Q v \quad [10]$$

where v can vary throughout space, but is steady at any given point. The minus sign indicates that a source is pulled by the stream in the opposite direction of v . (This equation [10] known as Lagally's formula is as important as Kutta-Joukovsky's formula for a flow with circulation.^{5, 27})

When the velocity potential corresponding to a source-sink distribution is known, the horizontal velocity is also known, and the resistance X can be written down as the integral of the product of the distribution and the horizontal velocity over the region of the distribution. The method can be generalized for calculating the mutual interaction between bodies (ships) advancing with constant speed in the same direction in tandem or for any other arrangement. The influence of fixed walls can be treated as a special case of this problem.

*Note the difference in the sign of the force due to charges of the same kind, when dealing with electric and with hydrodynamic phenomena.

A most important application was made by Dickmann to the problem of ship and propeller interaction.^{9,10,11} By replacing the latter (only when considering the mutual interaction of the system) by a sink or a sink distribution a comprehensive theory of thrust-deduction phenomena was developed.

Further interesting results can be obtained by applying this method to the calculation of forces and moments due to the motion of a wholly but not too deeply submerged body moving steadily parallel to the free surface; in principle this solution is given when the vertical velocity at any point of the image system representing the body is known.

4. WAVE RESISTANCE

Many curves representing wave resistance as a function of Froude number deduced from theory and experiment have been published; they are characterized by "humps" and "hollows" over definite ranges of speed-length ratios due to interference effects of different wave systems. The experimental curves reveal a smaller fluctuation than the theoretical because of the influence of viscous forces.

Various attempts have been made to calculate the wave resistance from the wave profiles; the most valuable proposal is due to R. Guilloton.¹⁷ A great difficulty in using wave profiles for resistance computation is the loss in accuracy due to difference errors. This difficulty is increased by the fact that the vertical pressure distribution due to waves does not follow closely the simple exponential law. We shall consider here the wave patterns only with the purpose of obtaining some general ideas about interference effects. The basic work on this subject is due to Wigley^{90,92,96} and Havelock.^{67,68,76}

A distinction must be made between a "wave" and "non-wave" portion of the profile; the latter does not contribute anything to the resistance in an ideal fluid and is, therefore, less important from the present viewpoint. Following Wigley we consider a cylinder (vessel of infinite depth) whose waterline is a parabola with some parallel middle body.

The wave part of the profile consists of:

- a. Wave systems due to the finite angle of entrance and run (the systems at bow and stern are identical when the angles are equal).
- b. Systems due to the curved parts of entrance and run.
- c. Systems due to the shoulders if these are pronounced.

However, the definition of such systems is made to a great extent by considerations of easy integration, so that different interpretations and divisions are possible.

The resistance curve can be split up into a monotonically increasing part and fluctuating components. Neglecting the parallel middle body, we obtain the resistance of the parabolic cylinder as the algebraic sum of 5 terms, due to:

- a. Bow and stern patterns (as if each existed alone).
- b. Curved sides (entrance and run).
- c. Interference of bow and stern.
- d. Interference of bow or stern with entrance or run.
- e. Interference of entrance and run.

Patterns a and b are not oscillatory, being proportional respectively to the 6th and 8th power of Froude's number F . For low values of F the finite angle at bow and stern is more important, but with increasing F the second term gains in value. Terms c, d, and e give fluctuating resistance curves.

The influence of the various terms depends on the speed and the form of the ship. Results obtained by Wigley for a prismatic pile with a trapezoidal half waterline differ widely from those corresponding to the parabolic waterline. The very pronounced shoulders (corners) in the former cause a strong interference effect between the bow and the shoulder system, while the influence of the shoulders is of secondary importance for the parabolic lines with parallel middle body.*^{90, 92}

Many discussions have been devoted to the length of separation on wave-making distance. Even the definition of this concept is not unique. Following D.W. Taylor,⁴² the most pronounced interference effects are due to the first crest just abaft the bow and the first crest of the sternwave system somewhat abaft the stern; hence, the distance between these crests may be considered as length of separation. On the other hand, in the opinion of the Froudes, this length should be defined as the distance between the bow crest and the trough caused by the after-shoulder. In the light of the preceding remarks and more detailed investigations by Wigley and Havelock, we quote the latter:

"Although simple empirical formulae for so-called wave-making distance may be of some use it is doubtful whether they are worth the time in inventing them, or in proving or disproving them...."⁹⁰

*When there are no corners in the waterline—as nearly always in actual practice—the concept of shoulder wave system becomes somewhat arbitrary.

Thus, it may be recommended to drop entirely the concept of wave-making length and to estimate the regions of high and low wave making, as functions of Froude number, from suitable experimental or theoretical curves. Particular, it should be emphasized that the so-called (P) "theory," based on similar reasoning and used to prophesy the positions of humps and hollows on the resistance curve, is at best an interpolation formula valid for a limited range of ship forms and Froude numbers,⁹⁶ and that the use of

$$(P) = \sqrt{\frac{2\pi}{\phi L g}} \times v \quad [11]$$

as abscissa when plotting resistance curves should be abandoned.

A great deal of confusion has been caused by substituting for the ship simple-pressure systems. Commonly, a positive and a negative pressure system are assumed to illustrate the action of the bow and the stern. This choice is based on the doubtful assumption that bow- and stern-wave patterns are similar in character except that crests and troughs are interchanged. The argument is not consistent, however, since the wave profiles due to the finite angle of stern and bow have the same sign, i.e., they both start with crests.

While two positive pressure systems are sometimes substituted for the ship, this also leads to an erroneous concept as the contributions of the curved sides of the hull to the wave pattern generated at the bow and stern have opposite signs. This is clearly brought out by Havelock⁶⁷ and Wigley,⁹⁰ whose work should be consulted by anybody interested in the subject.

4.1. CALCULATED AND MEASURED WAVE RESISTANCE

Various papers have been written with a similar purpose as the present one: a. To give a synopsis of the more important results obtained by the evaluation of existing theories, b. to describe the results of comparisons between theory and experiments, and c, to decide how far theory is able to help in the solution of practical problems. We begin with a synopsis of a recent publication by Giers and Sretensky¹³ which represents the most general attempt to answer the three points mentioned although we do not agree with the basic results reached.

Figures 4 and 5 have been recalculated from this paper.^{13*} They represent values of the coefficient $r = \frac{R}{\Delta}$ as a function of F , where R is the wave resistance calculated by Michell's integral for a family of elementary ships:

$$\eta = X(\xi) \cdot Z(\xi) \quad [12]$$

*The accuracy of the curves, Figures 4 and 5, may be sufficient for qualitative estimates only.

The dimensionless form depends upon a single parameter $C_p = \phi$, which appears as the parameter of the resistance curves $r = r(F)$. For all curves: $L = \text{const.}$ $B/H = 3$, $\beta = 0.95$. Two displacements, characterized by $\psi = \textcircled{M} = 8$, (Figure 4) and $\psi = 4$ (Figure 5), are investigated.

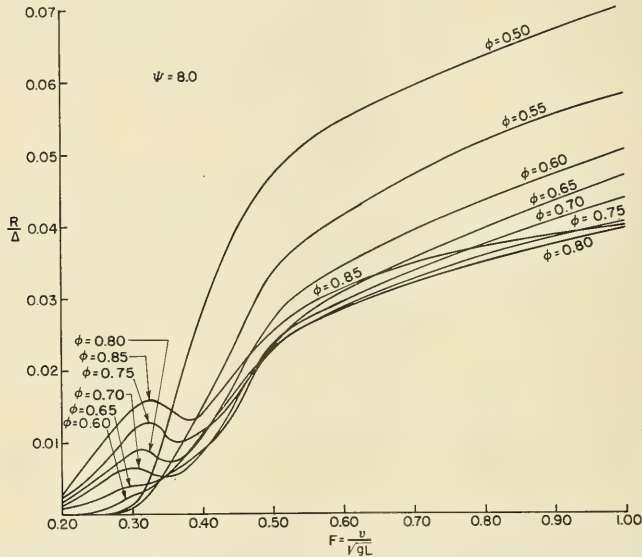


Figure 4 - Calculated Wave Resistance $\psi = 8$ (Sretensky)

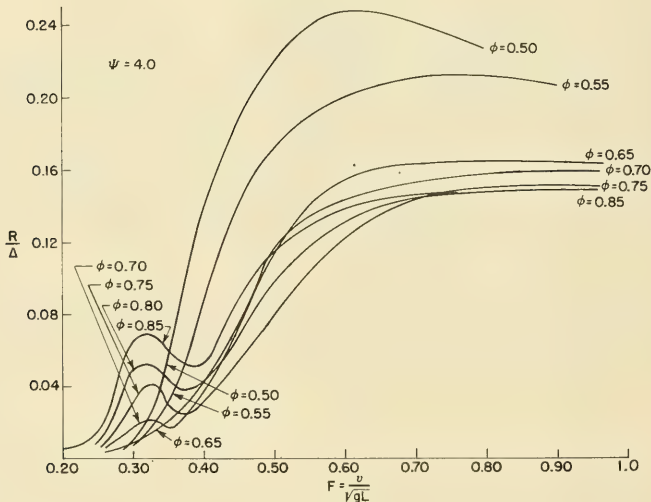


Figure 5 - Calculated Wave Resistance $\psi = 4$ (Sretensky)

Within any one diagram $\psi = \text{const.}$ L/B varies as $\sqrt{\phi}$ since, from the condition, ∇ is constant it follows that $\phi(B/L)^2$ is constant; the beam of the finer forms is larger than that of the fuller ones.

The curves " $\phi = \text{const.}$ " on the two diagrams 4 and 5 are connected by the relation $L/B = \psi^{3/2}$.

Although the equation for the hulls is so restricted that it can represent useful ship lines only within a limited range of prismatic and Froude numbers, and the representation of results is somewhat unfortunate, due to the use of ψ , the two diagrams give a useful general idea of resistance properties of ship forms over a wide region of prismatic and speeds.

A weak point of the investigation lies, however, in the deductions made from Figure 6, reproduced from the paper, which represents a comparison of the wave resistance calculated for the restricted mathematical lines and the residual resistance derived from Taylor's Standard Series. (For some unknown reason, another resistance coefficient

$$C_W = \frac{R}{\frac{\rho}{2} v^2 \nabla^{2/3}}$$

is used.) Thus the comparison is based on the equality of prismatic alone neglecting the difference between the actual shape or equations of the two series. This coarse approach yields a reasonable approximation within the region of the first and (in a lesser degree) of the second hump; it can, however, become inconsistent for lower Froude values. Checks indicate that in this region the calculated resistance of some of Sretensky's ill-chosen forms can be twice or three times as high as the calculated resistance of Taylor's models. Hence some important differences stated are essentially due to differences in forms used and have nothing to do with discrepancies between theory and facts.

Giers and Sretensky state: "The use of wave resistance coefficients calculated by theory appears to be inadmissible for drawing quantitative conclusions in the present state of knowledge." Especially, according to the author, these coefficients cannot be used for obtaining relations dependent upon the variation of one parameter while others are kept constant.

If these final conclusions—arrived at by one of the greatest authorities on the subject—were true, the theory of ship wave resistance would be useless from a point of view of naval architecture. We have shown that the conclusions are based on an erroneous procedure and cannot be upheld. More reliable, if less general, comparisons between calculated and measured resistances are due to Wigley and to an extent to the author, who came to much more favorable results.

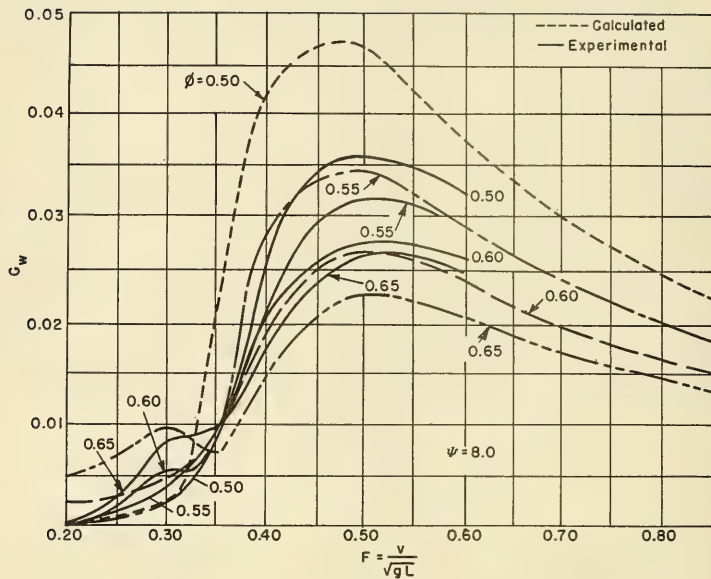


Figure 6a

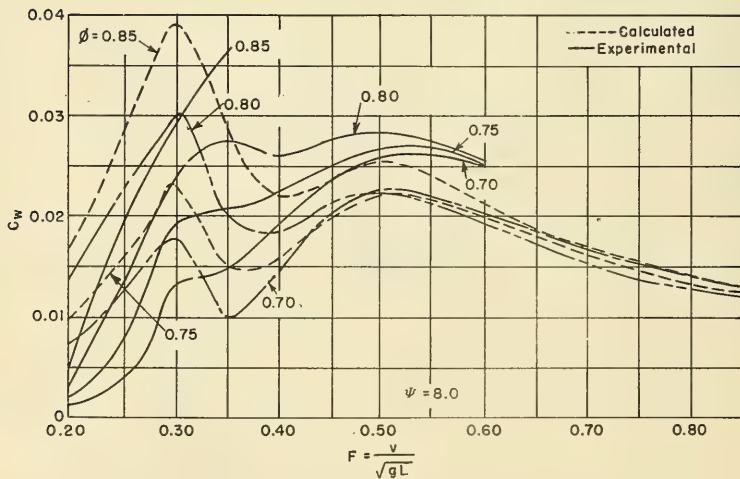


Figure 6b

Figure 6 - Comparison of Calculated and Measured Resistance (Sretensky)

The question may be raised as to why Sretensky's analysis based on the prismatic coefficient alone has failed, while the application of Taylor's Standard Series results, based on the same parameter, has proved to be extremely successful in practice except for full forms. The answer is that Taylor's forms are advantageous or reasonable; they have not been derived from a narrow family which may yield extremely bad forms as some admitted by Sretensky. A fortunate feature of Taylor's hull is for instance a small bulb, which improves resistance properties.

Indicative of the present state of knowledge is Figure 7—a typical set of resistance curves obtained by Wigley and reproduced by kind permission of the Institution of Naval Architects, from his paper presented before the INA in 1942.

In the region of the first hump* (at high Froude numbers) calculated values are less than the experimental ones (except for very low prismatic coefficients). According to Wigley this discrepancy is due to the fact that models and ships have freedom to rise and trim while calculations apply to a fixed condition.

At moderate (second and third humps) and low speeds the agreement in average absolute values depends upon the prismatic coefficient ϕ and L/B and B/H ratios. Theory overestimates the resistance of full and blunt bodies. The maxima and, to a still higher degree, the minima of the calculated curve are exaggerated; hollows in calculated curves generally appear flat in experimental curves. The very pronounced interference effects predicted by theory, especially those which result in extremely advantageous resistance properties, are thus smoothed out under actual conditions.

At very low Froude numbers, say $F \approx 0.18$, the measured residual resistance cannot be identified completely with the actual wave resistance. Presumably, for finer forms, the theoretical values are nearer to the truth than values based on "residual resistance."

There is a well known phase lag in the waviness of computed and experimental resistance curves: The humps of the latter appear at Froude numbers which are some 8 percent higher than predicted by theory (except the first hump). This shifting has been explained by the effects of viscosity; Wigley has shown that the phase lag is also slightly influenced by the size of models.

*The hump in the resistance coefficient curve (© curve) occurring at $F \approx 0.5$ is denoted by first hump, at $F \approx 0.3$ by second hump, etc. contrary to the habit in naval architecture, by which the hump at the highest speed is called "last" hump. This change appeared to be necessary since from a mathematical viewpoint we have an infinite number of humps between $0.5 \geq F \geq 0$.

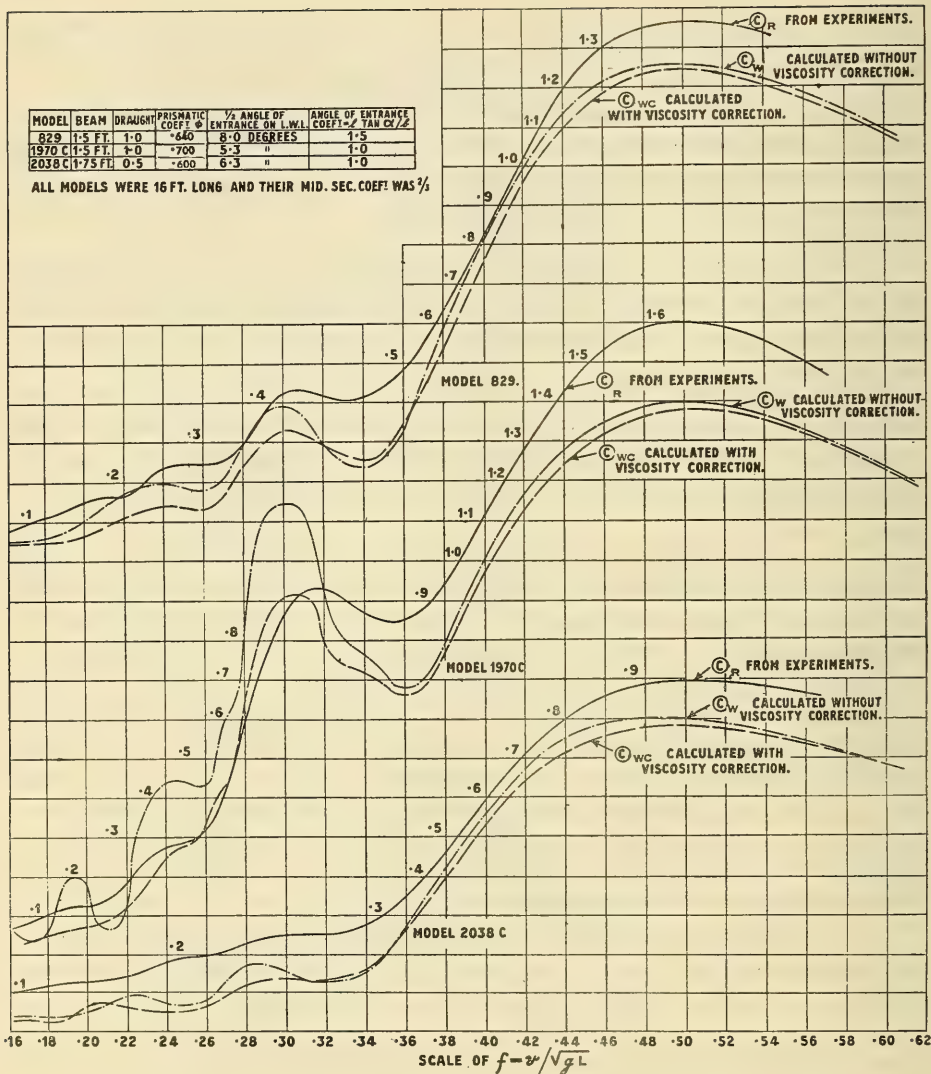


Figure 7 - Comparison of Calculated and Measured Resistance (Wigley)

(Reproduced by Permission of Institution of Naval Architects)

More detailed information will be given in the next chapters, which are intended to give a clear idea of what can and cannot be expected from the theory. Anticipating these results we assert that theory even in its present condition yields a powerful tool in resistance research and that an endeavor should be made to use it as a guide in developing closer approximations.

5. WAVE RESISTANCE AS A FUNCTION OF THE SHIP FORM

5.1. PROPORTIONS AND SHAPE

When treating Ship Forms 1.1.2 we made a fundamental distinction between the proportions of a hull and its dimensionless form. The discussion of wave resistance will be based on this division, since by it it appears possible to introduce a scheme into the manifoldness of dependencies of the resistance upon the form.

Therefore, our next task will be to estimate how far the resistance of a given form can be investigated independently of its proportions. The estimate will be performed by two steps: Within the validity of Michell's integral and outside of it.

The wave resistance formula denoted by Michell's integral is reproduced in Appendix 2. For our present investigation we substitute for the rather complicated integral a symbolic formula

$$R = \rho g \frac{B^2 H^2}{L} E\left(\frac{H}{L}, \eta, F\right) \quad [13]$$

In principle, as has been pointed out before, this expression which is equivalent to Michell's integral holds only for narrow wedgelike ships. The dimensionless factor $E(H/L, \eta, F)$ in which we are primarily interested is a complicated function of H/L , the dimensionless equation of the surface η and Froude number; it does not depend on the beam.

Thus we see at once that within the range of validity Michell's integral the dependence of the resistance upon the dimensionless form η is not influenced by the beam, and that no rigorous separation can be made between the influence of the surface equation η and the draft H (H/L).

Fortunately, the ratio H/L influences primarily the absolute value of R and only to a slight extent the character of interference effects. Thus comparison between different forms can be made for a definite H/L and the qualitative results (within the validity of Michell's integral!) safely extrapolated to other ratios of H/L . Havelock has shown that even calculations based on infinite draft give reliable results with respect to the position of humps

and hollows of the resistance curve.⁶⁶ Thus, using Michell's integral, the concept of dimensionless form seems to be quite appropriate. We shall go even a step farther by splitting up the resistance due to longitudinal and vertical displacement distribution and investigating them independently.

Matters become more complicated for wholly submerged bodies, but even here the character of the resistance curve is only slightly influenced for small changes in the depth of immersion.

It is a wholly different question how far the dimensionless form can be treated independently of proportions when dealing with actual ship forms to which Michell's theory does not rigorously apply. Let us put the question in a rather special but practical way: Can we assume that the merits of dimensionless forms tested at definite L/B and L/H ratios (for instance for slender bodies) remain the same for other ratios of L/B and L/H ? This problem involves a critical investigation into the validity of Michell's integral.

Lindblad²⁸ points out that the optimum shape of the sectional-area curve depends to some extent on the B/H ratio. D.W. Taylor's experiments with variants of the "YORKTOWN" indicate a slight dependence of the shape of the resistance curve upon B/H . More systematic work dealing with this problem is urgently needed.

A qualitative estimate can be obtained from a study of bodies of revolution. When their diameter/length ratio d/L is very small the sectional-area curve is proportional to the doublet distribution.⁴⁹ By reducing the speed of the generating uniform flow larger d/L ratios are obtained; dimensionless sectional-area curves are no longer affine to the distribution and do not coincide with each other. For a spheroid the deviation is characterized by a decrease in the "length of distribution" with increasing d/L , the prismatic obviously remaining constant $\phi = \frac{2}{3}$.

The following statement can be proved for a class of bodies of revolution: a. When the fineness coefficient of doublet distribution $\phi_d > 2/3$, the fineness coefficient of the resulting body ϕ decreases with increasing d/L (example: Rankine's oval), b. and when $\phi_d < 2/3$, the fineness coefficient increases with d/L . Thus one must infer that the image system suitable to generate a ship form depends on the ratio of principal dimensions. Hence, even the shape (and not just its magnitude) of a resistance curve derived for a dimensionless form varies in principle with these ratios.

Some additional information is given in the next chapter. We anticipate the conclusion: The assumption that the wave resistance of hulls can be treated independently as a function of the dimensionless form and the

proportions, means a rough if necessary approximation which must be corrected especially when the ship forms differ widely from "Michell's ship."

5.2. WAVE RESISTANCE AS A FUNCTION OF PRINCIPAL DIMENSIONS

If η is the equation of a hull and L is kept constant, the basic variations in beam and draft* are given by:

a. $H = \text{const.}$, B variable, i.e., an affine distortion in the direction of the y -axis.

b. $B = \text{const.}$, H variable, i.e., an affine distortion in the direction of the z -axis.

Two further variations are popular with experimenters:

c. $BH = \text{const.}$, i.e., an affine distortion along y and z .

d. $B/H = \text{const.}$, i.e., similarity distortion.

5.2.1. Variation of Beam for Constant Draft H

From Michell's integral or the symbolic expression [13], it follows immediately that the wave resistance varies with the square of the beam for all Froude numbers, thus

$$R = R(B) \sim B^2 \quad \text{or} \quad R \sim (B/H)^2 \quad \text{or} \quad R \sim (B/L)^2 \quad [14]$$

where H and L are constants. (As Michell's integral is valid only when $L/B \gg 1$, $2H/B \gg 1$ and $\frac{\partial y}{\partial x}, \frac{\partial y}{\partial z}$ are small, this simple result should be checked.)

The total of relevant experiments is astonishingly small; of these the most important measurements are due to Wigley.⁸⁴ He found that within the region $16 \geq L/B \geq 8$ agreement between theory and experiment is reasonably good, and that an exponent n somewhat smaller than 2 in the formula

$$R(B) > B^n \quad [15]$$

is more in keeping with experimental results.

From Taylor's experiment, an exponent $n \sim 1.6$ can be derived. An empirical curve $n = n(F)$ was given by Mumford⁵⁵ (Figure 9 on page 29).

Theoretical estimates of the validity of the law $R \sim B^2$ were made using limiting conditions, as follows:

a. Comparing a spheroid with an ellipsoid of twice the width it was found for $F = 0.226$ and $F = 0.50$ that the law $R \sim B^2$ holds within the accuracy of computation. A similar result was found using a theorem due to Lamb.²⁶

*Cf., Figure 8, page 28.

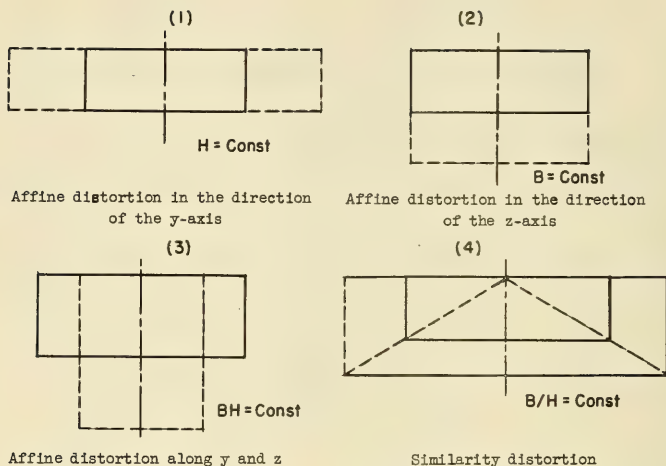


Figure 8 - Variations of Proportions

b. Another limiting condition may be derived from Hogner's resistance formula [18] for a pressure system. Using, for simplicity, a constant pressure over a rectangular region of the free surface, it was found that $n(F)$ in the formula

$$R \sim B^n(F) \quad [16]$$

is always smaller than 2. For example, $n(F) \cong 1.5$ for $F = 0.25$; in principle $n(F)$ depends upon B/L . Although this application is rather tentative, we may infer that for surface ships n is smaller than 2.

c. Finally, some information may be obtained from more advanced theories mentioned later, but no numerical computations have so far been performed.

Thus, the results of all the endeavors made to date is rather meager. However, from the known form of the solutions, we can infer that within the range of normal ship forms the wave resistance is a smooth function of B , and that small proportional changes in the beam cause correspondingly small changes in the wave resistance. When model experiments sometimes yield an abnormal result (as for instance in the case of the "NORMANDIE"), it is possible to assert that such a pathological behavior is not significant as to the wave resistance of the ship, whatever the reasons may be.

5.2.2. The Effect of Variation of Draft for Constant Beam B

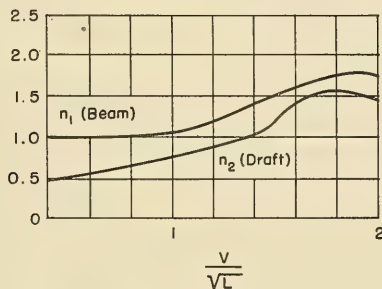
In this case the volume is proportional to draft.

Contrary to its dependence upon beam, the wave resistance cannot be expressed in an explicit manner as a function of the draft in a general way. However, approximate formulas can be derived from Michell's integral; they can be treated as special cases of the symbolic expression

$$R = \rho g \frac{B^2 H^2}{L} E\left(\frac{H}{L}, \eta, F\right) \quad [13]$$

Only a small number of such calculations has as yet been made.^{38,87,98} The form of the hull (η), especially the vertical distribution of displacement, influences to some extent the relation between resistance and draft, but even the longitudinal distribution can have some bearing on the problem.

For simplicity the influence of η on the function $E(H/L, \eta, F)$ can be neglected. Then E is a function of H/L and F only. Attempts have been made to approximate $H^2 E$ by a power relationship, $R \sim C H^{n(F)}$ where C is a constant [16] (Figure 9).



$$R \sim B^{n_1} H^{n_2}$$

Figure 9 - Mumford's Exponent Curves

Obviously, such a simple approximation can be expected to hold only for a limited range of H/L which again is dependent upon F .^{*} Some values of n are given below:

^{*}In the limiting case of $H/L \rightarrow 0$, a quadratic law $n = 2$ results as an asymptotic value, which, however, is of no practical use because of the breakdown of the theory.

a. From resistance curves calculated by Sretensky⁸⁸

1. $n \approx 1.4$ for $0.30 > F > 0.235$, and

2. $n \approx 1.6$ for higher Froude numbers $0.7 > F > 0.47$; both are valid in the neighborhood of $H/L = 1/20$.

b. Using Wigley's paper⁸⁷ the following data are obtained:

1. $n \approx 0.85$ for $0.23 > F > 0.18$,

2. $n \approx 1.3$ to 1.5 for the second hump $0.32 > F > 0.26$.

3. $n \approx 1.3$ to 1.7 for the first hump $0.6 > F > 0.36$.

It seems that on the average the exponent n is higher for finer ship forms. In Wigley's case the range of H/L is

$$\frac{1}{16} \geq \frac{H}{L} \geq \frac{1}{32}$$

Mumford's curve agrees reasonably with theoretical computations for the first hump; in the range of lower Froude numbers empirical values are smaller than the theoretical ones.

Using Michell's integral, both of the following cases, c and d, can be easily derived from the two basic ones (a and b).

c. A number of well known experiments can be classified under the conditions: $BH = \text{const}$, $A_M = \text{const}$ and displacement constant.

These are the sets of Taylor's Standard Series⁴² $B/H = 2.25$ and $B/H = 3.75$, Ackerson's Series,¹ Rota's,⁴¹ and Kent's⁵⁷ experiments.

Obviously, the wave resistance R increases with increasing B/H , as the beam contributes more to the drag than the draft, although the total resistance may change only slowly within a length/beam ratio $10 \geq L/B \geq 8$. No theoretical analysis has been applied to this case.

d. In this case $B/H = \text{const}$; L/B and L/H are variable (similarity transformation) and displacement varies with B^2 . This problem has been studied most thoroughly by D.W. Taylor,⁴² Ackerson,¹ and Bragg.⁶ The displacement-length ratios of the Taylor Standard Series models were varied by similarity transformations. A comparison has been made between calculated and measured resistances for sets of models defined analytically and run in the Berlin Tank.⁹⁸

The wave resistance grows very rapidly with increasing B , and theoretically with the fourth power for extreme values of F . For $L/B = 6$ the theoretical values are excessive.

In agreement with Wigley's conclusion as to the dependence of the resistance upon beam, we infer that theory overestimates the absolute value for low values of L/B and relatively high prismatics at moderate Froude numbers (below the first hump).

Figures 10, 11, and 12 show an attempted comparison by Antimionoff of calculated and measured resistance coefficients

$$C_W = \frac{R_W}{\frac{\rho}{2} \nabla^{2/3} U^2}$$

as functions of L/B and B/H .

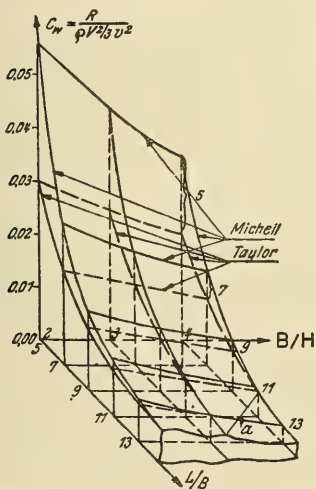


Figure 10 - Calculated and Measured Resistance as Functions of Principal Dimensions

The analytical ship forms for which the computations were made do not correspond very closely to those of the models tested. Because the theoretical values were primarily based on the present author's work

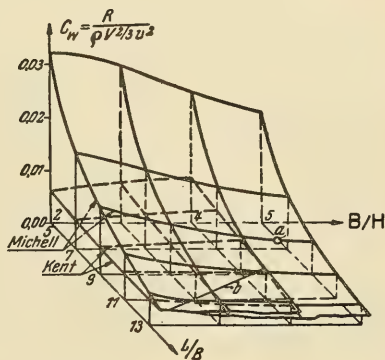


Figure 11 - Calculated and Measured Resistance as Functions of Principal Dimensions

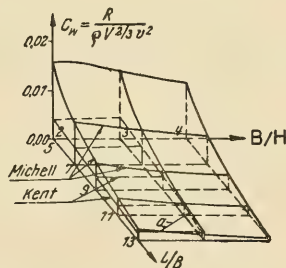


Figure 12 - Calculated and Measured Resistance as Functions of Principal Dimensions

while the model results were taken from those of Taylor⁴² and Kent,⁵⁷ only ϕ and β were identical. This coarse procedure may invalidate the comparison of absolute values for moderate and low Froude numbers but not the general trend as a function of the principal dimensions.

For $F = 0.387$ (Figure 10) the agreement between theory and experiment is good as to the character of the curves and reasonable with respect to absolute values above an L/B of about 8.

Figure 11, valid for $F = 0.24$, shows a complete failure of the comparison. Allowing for the difference in forms mentioned above, it can be said that:

a. The values of the calculated wave resistance are much exaggerated for smaller L/B ratios.

b. Even the trend in the experimental residual-resistance curves and calculated wave-resistance curves as functions of B/H does not agree. This indicates that the "residual resistance" does not furnish any information about the actual wave phenomena in the present case because of the presence of viscous drag. An increase in this resistance with increasing B/H has been found by experiments with double models.¹⁴

c. The diagram (Figure 12) representing conditions for slow vessels does not show such pronounced anomalies, but supports the impression that the model results in question do not contribute to the analysis of wave resistance.

To summarize, we may say that theory has contributed some rough estimates of the relations between principal dimensions and wave resistance; their validity is limited mainly by the L/B ratio. Only a small number of experiments exist which are reliable enough to check the theory and to deduce simple empirical laws for the basic cases discussed. The presence of viscous-form drag and other viscosity effects have so far seriously hampered the study of wave resistance of slow full ships.

5.3. THE WAVE RESISTANCE AS A FUNCTION OF THE HULL SHAPE

5.3.1. General Remarks

The restrictions under which the concept of dimensionless form can be used in resistance research have already been enumerated. The present task is: a) To analyze theoretically the resistance properties of different forms, b) to deduce some general rules from this analysis, and c) to report on experimental checks. The influence of the longitudinal distribution of displacement on resistance is the most important problem, both from the viewpoint of theory and practice.

It was found that in principle resistance effects due to the vertical and longitudinal distribution of displacement cannot be separated; fortunately, however, it is possible to derive a great number of characteristic properties of ship forms without considering the shape of sections and the draft, which together determine the vertical distribution.

In the light of this knowledge, the basic item is the sectional-area curve; it is therefore natural to begin with the well studied elementary shapes given by

$$\eta = X(\xi) Z(\zeta) \quad [12]$$

which embody a given sectional-area curve in the simplest way. Later, more complicated forms will be investigated.

The present most important chapter of the paper is performed presented in a rather rudimentary state, since numerous lengthy computations needed for a rigorous discussion are not yet completed. Thus, instead of calculated resistance curves some simple intermediate functions are used as a basis of our analysis.

This analysis uses systematic geometrical variations of ship lines. In principle, a more elegant way would consist in minimizing the resistance integral. The latter procedure yields forms of least resistance which are basic results in themselves. Besides, when trying to establish simplified useful relations between important form parameters and resistance properties it is generally favorable to investigate good forms, since because of the possibly complicated actual relations the results may depend upon the selection of the forms. However, the method of minimization has not yet lead to a satisfactory solution; hence, the approach of geometrical variation adopted here appears to be the only one suitable.

It is pertinent to mention here why so many "mathematical models" have been tested which bear little resemblance to actual ship forms, since the erroneous opinion is widely held that theory can deal only with oversimplified models:

a. Simplicity of mathematical expressions was aimed at in earlier work when methods of computation were not developed and the physical bearing of the theory was unknown.

b. Variations in form for basic research must sometimes be made not with the purpose of improving forms, but of obtaining pronounced changes in resistance.

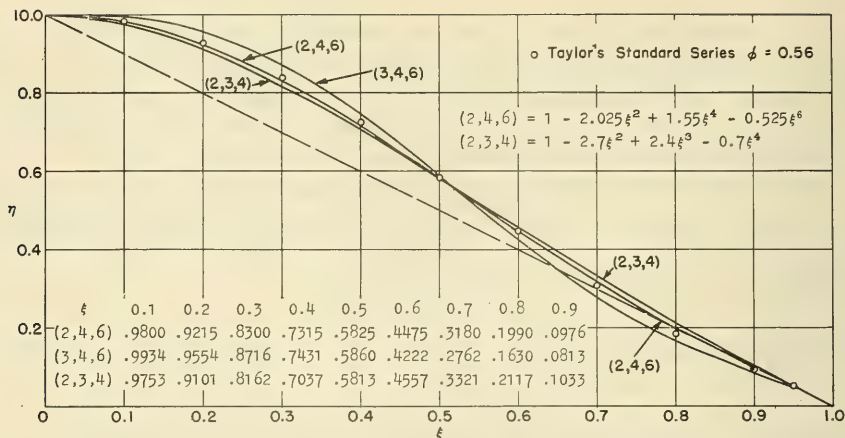


Figure 13 - Sectional Area Curves of Models Berlin 1337, 1370
Expressed by the Equations (2,3,4; 0.56;1) (2,4,6; 0.56;1)

Various attempts have been made, using Michell's integral, to perform wave-resistance calculations for "actual" ship forms, i.e., forms defined graphically by means of the normal-lines plan. No advantage is gained by such methods,* as it is easier to "mathematize" the ship form and then perform calculations. This leads to an easier method of comparing forms and improving their resistance properties, which in the present state of the theory is a much more valuable achievement than the possibility of calculating the resistance for an individual form represented graphically.

To get an idea of what actually can be reached by application of theory we refer to Figure 7 on page 24, discussed earlier, and to Figure 14 where the differences between the resistances of two models obtained from experiment and theory are compared. The results are impressive. In the light of these and other investigations it is impossible to question the practical value of the present theory.

5.3.2. Longitudinal Distribution of Displacement

Outline of a General Procedure: A survey of the resistance properties of ship forms must be based upon sufficiently general equations; the failure by using forms with a single arbitrary parameter has already been discussed. Basic families of ship lines studied earlier admit of sufficient variations (see Figures 15 and 16). Although each set contains only two parameters ϕ , t , a third one can be introduced by immediately "mixing" two families or by adding an appropriate polynomial.

*This criticism does not apply to a method proposed by Guilloton.

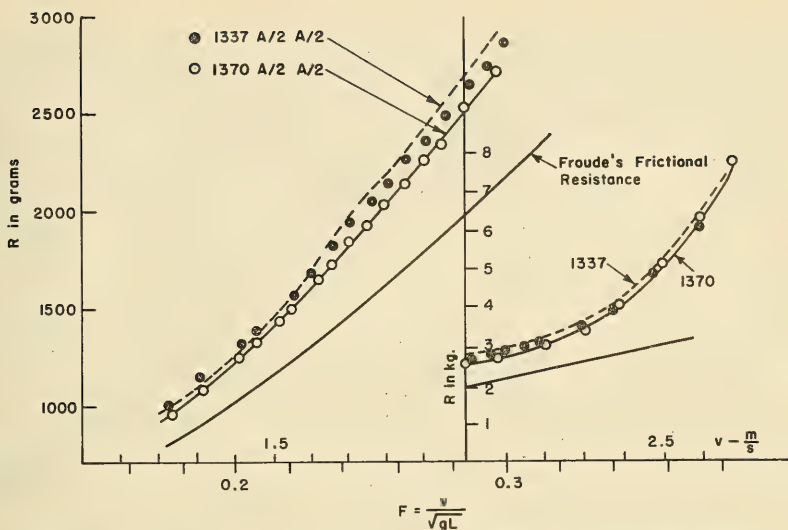


Figure 14 - Models Berlin 1337, 1370, Curves of Total Resistance

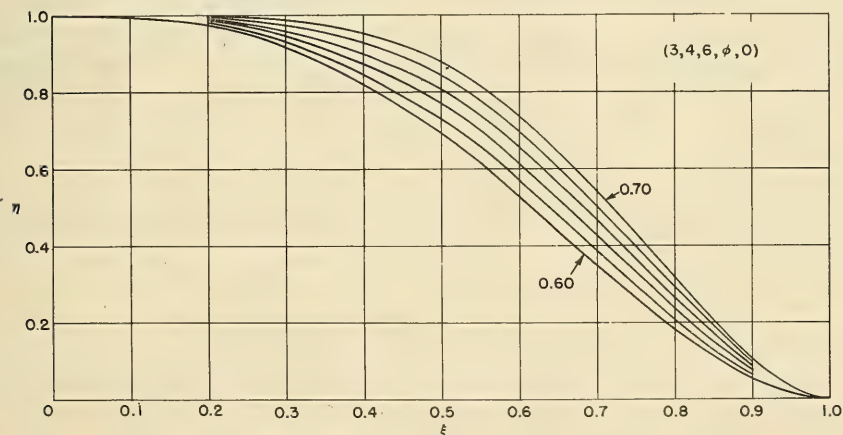


Figure 15 - Ship Lines, Basic Family $(3,4,6; \phi; t); t = 0$

Taylor's curvature value κ can serve as such a third parameter. It will be shown that it is a useful concept, but that even three parameters may not be sufficient to fix the resistance properties of ship lines.

The process of "mixing" or adding form polynomials can be extended resulting in a polynomial with an increasing number of powers and arbitrary coefficients. In addition, a parallel middle body can be inserted or a bulb

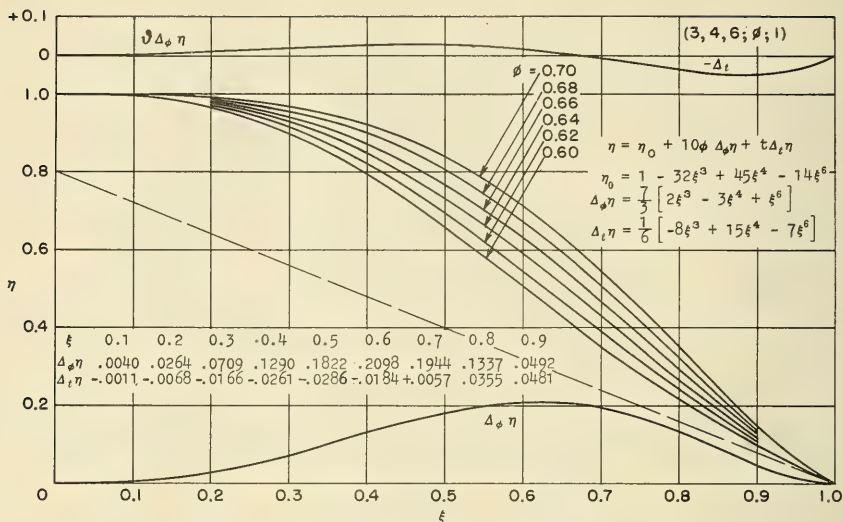


Figure 16 - Ship Lines, Basic Family (3,4,6; ϕ, t) $t = 1$

added. Thus various form parameters mentioned earlier (page 6) can be introduced; new parameters may be found by analyzing calculated resistance curves.

The calculation of wave resistance for sets of lines which cover the whole range involves a considerable but not prohibitive amount of work. The procedure adopted is outlined below.

As the first step, elementary ships symmetrical with respect to the midship section are analyzed. Further steps will deal with asymmetry and general forms of hulls.

In the equation

$$\eta = X(\xi) Z(\zeta)$$

let $X(\xi)$ represent the longitudinal distribution of displacement and contain a great number of members with arbitrary parameters, $Z(\zeta)$ may be of simpler form.

Intermediate functions based on the slope $\frac{\partial X(\xi)}{\partial \xi}$ are calculated: To each power ξ^n in $\frac{\partial X(\xi)}{\partial \xi}$ corresponds a function

$$M_n(\gamma) = \int_0^1 \xi^n \sin \gamma \xi d\xi \quad [17]$$

Grouping products of these functions $M(\gamma)$ in an appropriate manner and performing a further integration of these products multiplied by some other factors, some fundamental values are obtained and tabulated. From

these tables the wave resistance of practically any assumed normal elementary form with simple midship section can be derived by multiplication with the parameters involved and subsequent summation. A first contribution in this direction has been made by Wigley:⁸⁷ He performed calculations for the family (2,4,6, ϕ ,t) based on a slightly different form of the polynomial. Although the author has himself used this type of equation in earlier work, he prefers the expanded form

$$\eta = 1 - a_2 \xi^2 - a_4 \xi^4 - (1 - a_2 - a_4) \xi^6 \quad [18]$$

or in Taylor's notation

$$\eta = f_0(\xi) + \alpha f_1(\xi) + t f_2(\xi) \quad [19]$$

Wigley's work covers only one basic family of ship lines, but it includes a check of calculations by experiments.

Method of Approximate Discussion: Pending the computation of tables of complete resistance integrals, a simple if rather coarse procedure has been developed which allows comparison of the relative merits of forms. It is based on the discussion of the integrand of Michell's formula, or rather on only one part of it, which depends on the longitudinal displacement distribution and is formally handled by the $M_n(\gamma)$ functions previously mentioned.

This integral is written in the symbolic form

$$R = C \int_0^\infty S^2(\gamma) \phi^2(\gamma) f(\gamma) d\gamma \quad [20]$$

where γ , the variable of integration varies for a given Froude number F between $\gamma_0 = \frac{1}{2F^2}$ and infinity,

$f(\gamma)$ is a simple algebraic function,

$\phi^2(\gamma)$ is a function dependent upon the vertical distribution of displacement, and the product

$\phi^2(\gamma)f(\gamma)$ ensures a rapid decrease of the integrand with increasing γ .

$$S(\gamma) = \int_{-1}^{+1} \frac{\partial X}{\partial \xi} \sin \gamma \xi d\xi \quad [21]$$

is an oscillating function dependent upon the longitudinal displacement distribution. $S^2(\gamma) \geq 0$ represents the most important factor of the integrand.

The longitudinal function $S(\gamma)$ can be easily computed from tabulated intermediate functions $M(\gamma)$ for a given $X(\xi)$.

Basic families of $S(\gamma)$ or $S^2(\gamma)$ functions are plotted in the same way as families of ship lines, with which they are associated, for instance by keeping $t = \text{const}$ and varying ϕ by steps of 0.02. (Figures 24 to 28) $S(\gamma)$ is a linear function of ϕ , t , when the associated ship line (water line) is also a linear function of ϕ , t . Corresponding to difference curves for the ordinates $(n_1 n_2 n_3; 0.1, 0)$ and $(n_1 n_2 n_3; 0; 1)$, difference curves of $S(\gamma)$ can be plotted which we denote by $\Delta_\phi S(\gamma)$ and $\Delta_t S(\gamma)$ * These permit a whole set of $S(\gamma)$ curves to be developed for various ϕ and t values, when one curve $S(\gamma)$ for ϕ_0 , t_0 is known; they are also a valuable help in various discussions.

Using the squared values $S^2(\gamma)$, the resistance of different ship lines can be compared as follows for a given Froude number F :

a. Calculate $\gamma_0 = \frac{1}{2F^2}$. Only $S^2(\gamma)$ values to the right of γ_0 are to be considered for a given F .

b. The area enclosed between the $S^2(\gamma)$ curve (multiplied by the monotonic decreasing function $\phi(\gamma) f(\gamma)$), the ordinate at γ_0 and the γ -axis is proportional to the resistance. Taking into account all conditions, generally the first waves of the $S^2(\gamma)$ are decisive for the determination of resistance. For a first orientation, comparisons can be made without multiplying $S^2(\gamma)$ by $\phi^2(\gamma) f(\gamma)$.

c. Values of γ_0 for which $S(\gamma)$ or $S^2(\gamma)$ are close to zero characterize the position of a hollow in the resistance curve, a γ_0 value just to the left of a crest of the $S(\gamma)$ or $S^2(\gamma)$ lines, corresponds to a hump. Thus we can estimate regions of low and high resistance by simple inspection.

Results of Investigations on Longitudinal Distribution of Displacement: D W. Taylor's experiments have revealed fundamental relations between the resistance and the standard parameters ϕ , t .** It has been shown that theory has succeeded in obtaining results which agree closely with these experimental data. The following discussions will, therefore, deal preferably with more refined form effects which are not so universally known. The success of these investigations depends to a high degree upon the use of mathematical lines.

*Some examples of these curves can be derived from Figures 24 and 25.

**Exceptions will be mentioned later.

Let us study the $S(\gamma)$ function for a given polynomial, say $(2,4,6; \phi; t)$ Figure 24 and $(3,4,6; \phi; t)$ Figure 25; the parameter $t = t_0 = \text{const}$. Curves for different prismatics intersect at fixed points; the abscissas of these points are given by zero values of the curves $\Delta_\phi S(\gamma)$, and are constants for a basic family; the ordinates vary with varying t , but are constants for $t = t_0 = \text{const}$.

When all maxima of an $S(\gamma)$ curve coincide with such fixed points, we obviously obtain a curve with small amplitudes and good resistance qualities over the whole range of Froude numbers. When such a coincidence occurs only for one or several points the particular curve may be advantageous over a limited speed range. For $t_0 = \text{const}$ and a given Froude number F there is a value of the prismatic which corresponds to a minimum resistance; a further reduction of ϕ means a deterioration of the resistance properties. This agrees with experiments. From Taylor's Standard Series the prismatic of least specific resistance appears to be something like 0.52 for moderate Froude numbers (second hump); following theory for hollow forms with $t = 0$, the minimum is somewhat lower as can be easily inferred from the $S(\gamma)$ curves.

We investigate now the influence of the curvature parameter K on the resistance. We compare for this purpose the family $(2,4,6; \phi; 1)$ with the related $(2,3,4; \phi; 1)$, keeping for simplicity the tangent value t constant and equal to one. The curvature at the midsection is $K = -2a_2$.

$$\eta = (2, 3, 4; 0.6; 1) = (2, 4, 6; 0.6; 1) = 1 - 1.5\xi^2 + 0.5\xi^4 \quad [22]$$

is common to both families. If we let $\phi < 0.6$, it can be shown that for equal values of ϕ the coefficient a_2 is higher in $(2,3,4)$ than in $(2,4,6)$, hence the form $(2,3,4,)$ has, ceteris paribus, a higher curvature at the midsection. When $\phi > 0.6$, the reverse holds.

Take the curves corresponding to $\phi = 0.56$; it is seen from Figure 13 that they differ only slightly. However, the functions $S(\gamma)$ or $S^2(\gamma)$ differ very much in the region $11 > \gamma > 8$, and we must expect that the resistance associated with $(2,4,6; 0.56; 1)$ will be much lower over an extended range of Froude numbers.

A crucial test with rather wide consequences therefore appeared to be possible; it was performed at the Berlin Tank¹⁰⁷ and yielded a beautiful agreement between calculation and experiment, see Figure 14. Model 1370 was developed from the sectional-area curve $(2,4,6; 0.56; 1)$ and Model 1337 $(2,3,4; 0.56; 1)$. Unfortunately, because of a widely spread but ill advised thrift, the original readings have not been published, but the author testifies that every effort was made to obtain reliable results. Furthermore, the results

were indirectly checked later by fitting equal bulbs to the two models. Further comparative calculations were made for sectional-area curves of Taylor's Standard Series.⁸⁹ From Figure 14 it follows that Taylor's form for $\phi = 0.56$ is very near to the good one (2,4,6; 0.56; 1). The associated $S(\gamma)$ and $S^2(\gamma)$ curves indicate clearly the excellent resistance properties of Taylor's forms for low and moderate Froude numbers, when $\phi = 0.52$ and 0.56 .

Important conclusions can be drawn from these investigations, which have been anticipated to some extent in our introductory remarks:

a. Small deviations in form may cause appreciable differences in wave resistance; the "fairness of lines" does not give the slightest indication as to wave-resistance properties. For instance the line (2,3,4; 0.56; 1) is "fairer" than (2,4,6; 0.56; 1) since it has only one point of inflection.

b. Every ship form is a unique problem; one must be very cautious in extra- or interpolating resistance properties when the decisive parameters of the problem are not known.

c. Theory gives powerful, if not thoroughly reliable, means of investigating even fine peculiarities of form with respect to their wave-resistance properties.

d. Changes in wave resistance due to deformations of models (for instance of wax models by high temperature) may account in some cases for inconsistencies in model results, especially in cases where repeated experiments do not agree with the original ones.

In the present case the parameter K proved to be significant as regards wave resistance. As an example, the resistance properties of two ship lines were investigated, which were derived by adding to a given line two polynomials expressed symbolically by (2,3,4,6; 0; 0) and (2,4,6,8; 0; 0) Appendix formulas [37] and [38]. The resulting lines have the same parameters ϕ , t , K ; nevertheless the resistance functions $S(\gamma)$ differ appreciably. Otherwise expressed, in such cases it is not possible to fix the resistance properties of lines even by three parameters.

The two families discussed characterized by $K < 0$ are typical for hulls run at $F < 0.25$. The absence of a parallel middle body is an important feature.

An inspection of the $S(\gamma)$ or $S^2(\gamma)$ curves, Figure 24, page 45, explains formally why in the ascending branch of the first hump higher prismatics and t values are beneficial: Both tend to shift the steep rise of the curves to the left towards higher Froude numbers. This property of shifting is valid for smaller Froude numbers too, but here the effect mentioned is canceled by others.

Low prismatics ϕ are a necessary condition for good resistance qualities below the first hump, $F < 0.32$. Normally considerations of resistance are so decisive for vessels operated within the range $0.32 > F > 0.25$ that only small ϕ values should be used. Grave mistakes have been committed when designing liners with rather high prismatics, although the basic facts could have been easily ascertained from Taylor's Standard Series. For ship types run at $F < 0.25$, it may be more advantageous to compromise between resistance and carrying capacity, since the absolute value of the wave resistance decreases. Thus, moderate prismatics become a reasonable solution.

At the same time, ship lines with vanishing curvature K gain in importance, as may be guessed from Figures 26 to 28 representing $S(\gamma)$ curves for ϕ and $t = \text{const}$ corresponding to different equations. The family (3,4,6) appears to be useful over a certain range; with increasing ϕ and reduced speed ratios, ship lines corresponding to higher degree polynomials become advantageous. The influence of $K = 0$ or, more generally, of a parallel middle body expressed by an increase of degree in a polynomial on the resistance is opposite to the increase of ϕ and t : The humps in the function $S(\gamma)$ are shifted to the right (smaller F). Our analysis shows that Taylor's Standard forms $\phi = 0.6$ and $\phi = 0.64$ are quite successful, but not outstanding within their useful speed ranges; it is possible to obtain better results when bulbs are fitted to some of the good normal forms here discussed (Figures 19, 20, 26, 27).

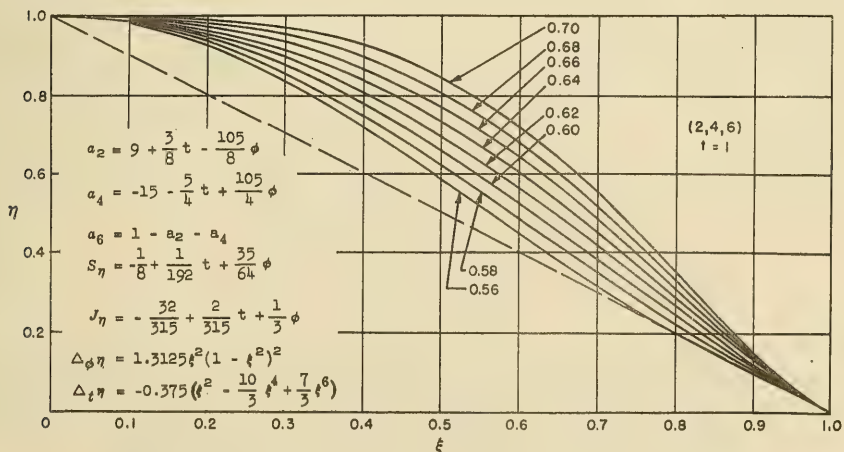


Figure 17 - Ship Lines - Basic Family (2,4,6; ϕ, t) $t = 1$

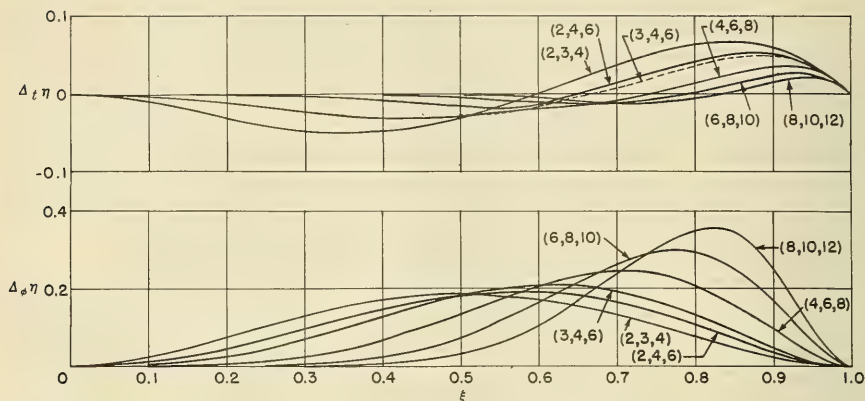


Figure 18 - Polynomials $(n_1, n_2, n_3; 0.1; 0)$ and $(n_1, n_2, n_3; 0; 1)$

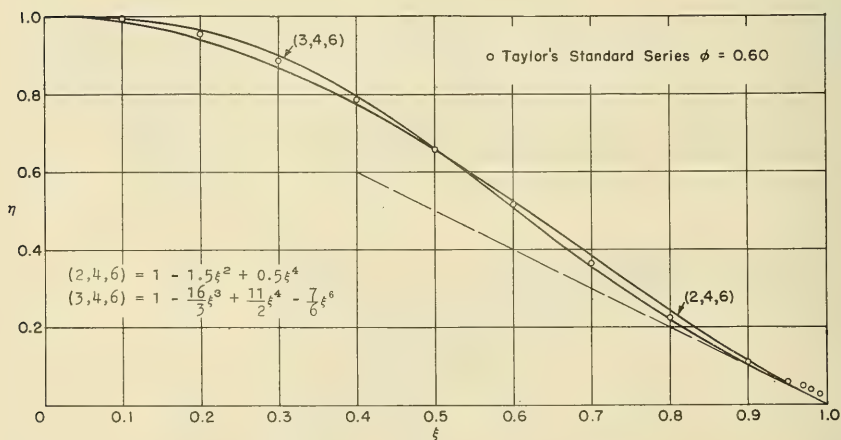


Figure 19 - Examples of Lines

$\phi = 0.60, t = 1$

An example has been given of how to find an economical lower limit of prismatics from resistance considerations (page 39). Still more important is the problem of finding a corresponding upper limit for a given Froude number. Discussing the family (4,6,8) the result was obtained that for $F = 0.25$, $\phi = 0.68$ is a reasonable value which cannot be exceeded without loss in efficiency (Figures 21 and 28). To check the deductions the curve (4,6,8; 0.68; 2) was used as sectional-area curve of the forebody in a model; the results of

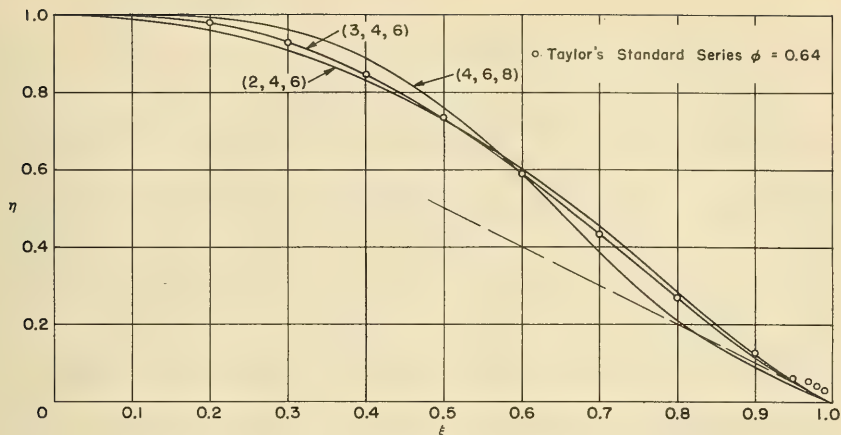


Figure 20 - Examples of Lines

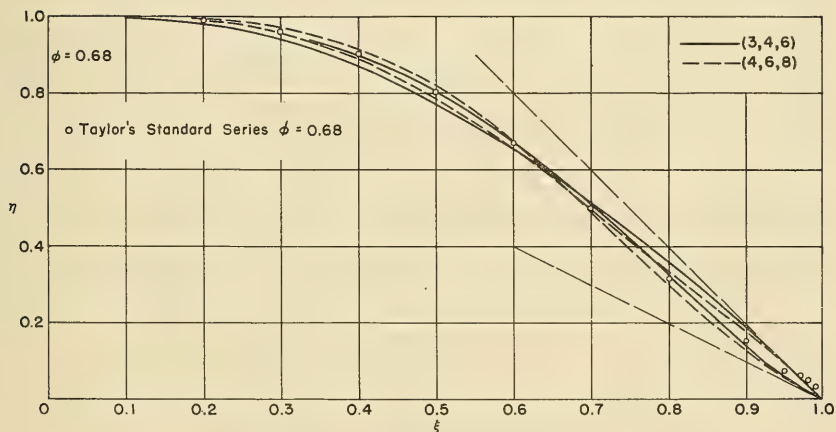
 $\phi = 0.64, t = 1$ 

Figure 21 - Examples of Lines

 $\phi = 0.68, t = 1, t = 2$

towing experiments with this model supported calculations in a very satisfactory manner.¹⁰⁶ The shape of the curve denoted as "small swanneck" is closely related to a bulb form. However, the "small swanneck" remained superior to normal lines fitted with a bulb; it is definitely superior to Taylor's Standard form for $\phi = 0.68$. We dwell at some length on this range of speed since it acquires increased interest in connection with fast cargo ships.

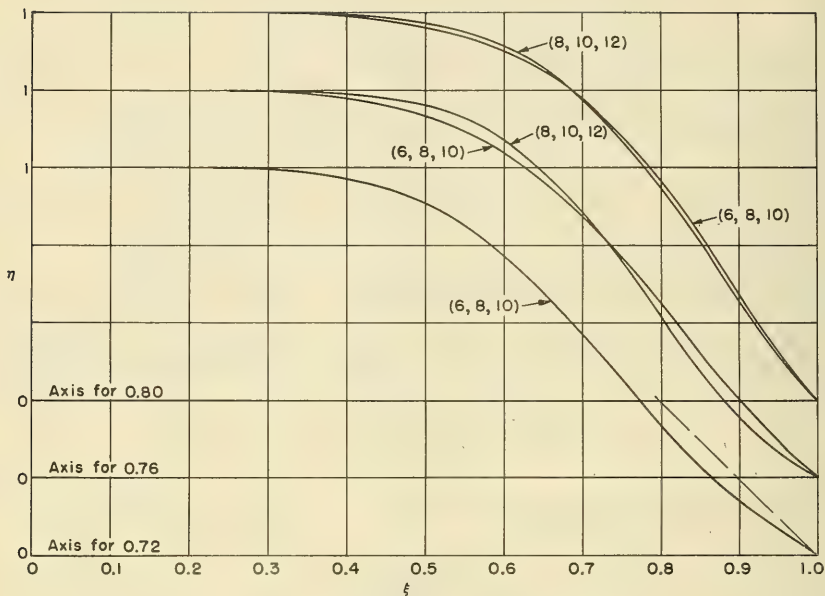
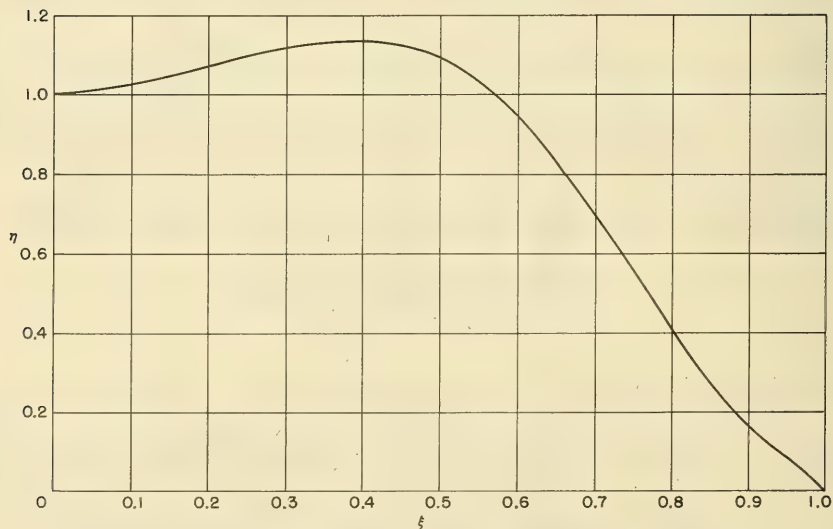


Figure 22 - Examples of Full Lines

Figure 23 - "Pathological" Sectional-Area Curve $\phi = 0.82$, $t = 2$

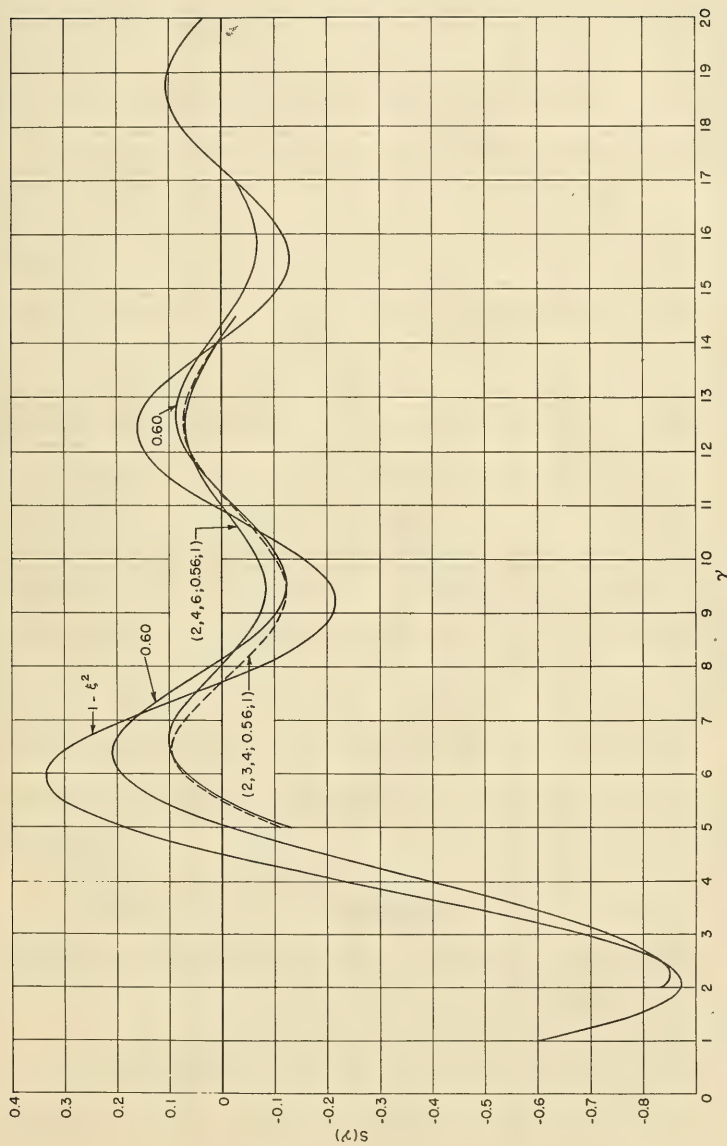


Figure 24 - Resistance Function $S(\gamma)$ Corresponding to Some Lines of the Family $(2, 4, 6)$ and to $(2, 3, 4; 0.56; 1)$

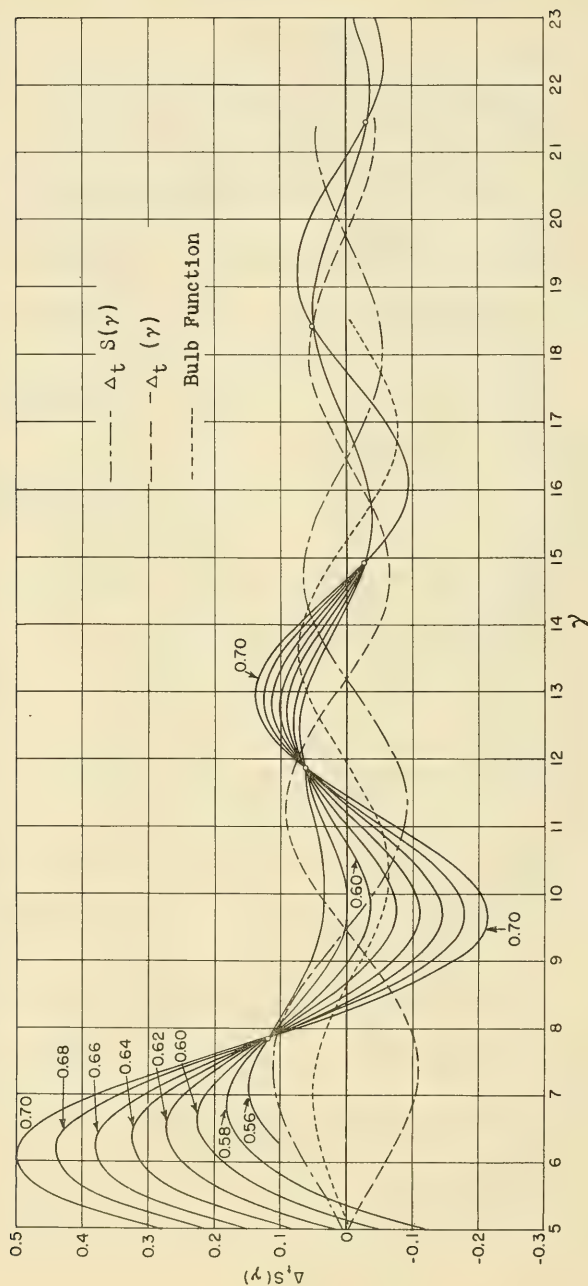


Figure 25 - $S(\gamma)$ for $(3, 4, 6; \phi; 1)$ and a Bulb Polynomial

The difference between the continuous lines and the Bulb function is the $S(\gamma)$ function for the corresponding hull forms with bulb.

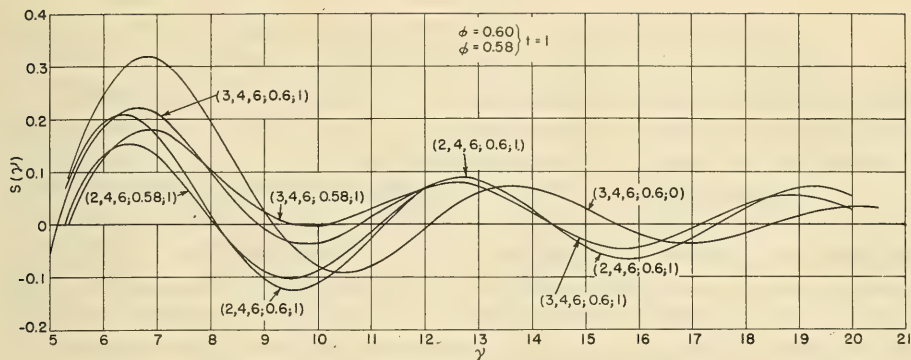


Figure 26 - $S(\gamma)$ for Some Lines $\phi = 0.60$

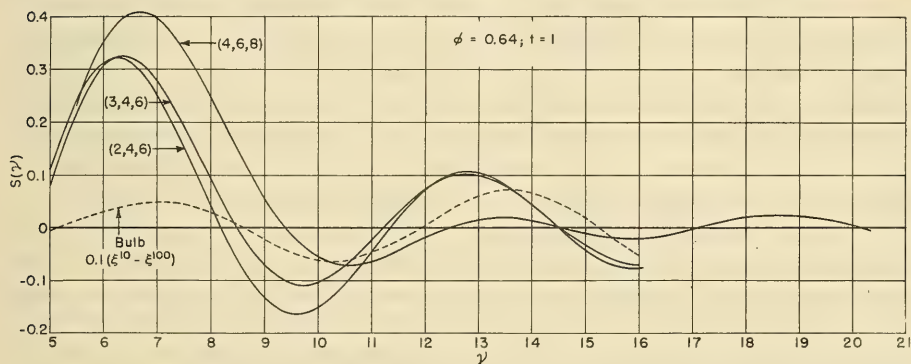


Figure 27 - $S(\gamma)$ for Some Lines $\phi = 0.64$

Extended investigations were made on fuller forms following the same lines. However, less success was reached with respect to the applicability of results;⁹⁸ Figure 22.

We were not able to develop forms with prismatics of 0.72, which have actually a low resistance at a Froude number above about 0.22. Up to this limit an equation (6,8,10; 0.72; 2) yields theoretically excellent resistance qualities; unfortunately, experiments have not corroborated the calculations.

The degree of the polynomials used (sectional-area curves) which is indicative of the amount of parallel middle body, is sometimes more decisive than the choice of t and, within narrower limits, even ϕ . Thus, it can happen that for low Froude numbers the resistance may not be very sensitive to changes in ϕ provided an upper limit is not exceeded.

With decreasing F , the optimum t values decrease. Also on the average, it depends upon the prismatic and the degree of polynomial used. While very fine ships yield good results with $t = 0$ up to $F = 0.26$, full vessels need t values > 0 for $F = 0.18$ and even less. Thus, charts indicating optimum t values must be prepared for several prismatics.

Absolute values of calculated resistance are not reliable when dealing with high ϕ and low F ; the results due to pronounced interference effects are especially doubtful. Investigations on a pathological model, Figure 23—whose resistance qualities according to theory should be outstanding—are very significant: Experiments did not agree at all with these deductions up to a Froude number of about 0.19, but for some higher values of F the form was efficient.

Thus, even a qualitative agreement is sometimes lacking when dealing with full hulls run at speeds below F about 0.19. For this reason, a more thorough discussion of this most important subject is delayed until further research has been done.^{63, 99, 106} This work must be based on a closer investigation of the different parts of wave phenomena constituting the total wave resistance, of wave patterns due to bow and stern angles and curved parts and the mutual interference of the systems mentioned. No decisive attempt has been made so far because of the lack of appropriate tabulated functions.

The following suggestion may be of interest to the experimenter when investigating the frictional resistance of plates. It is well known that, at some Froude numbers, wave phenomena can influence results. Havelock has given a first estimate on the subject⁷⁷ using a parabolic form and an arbitrary value for the width of the plate. A closer approximation is reached when the

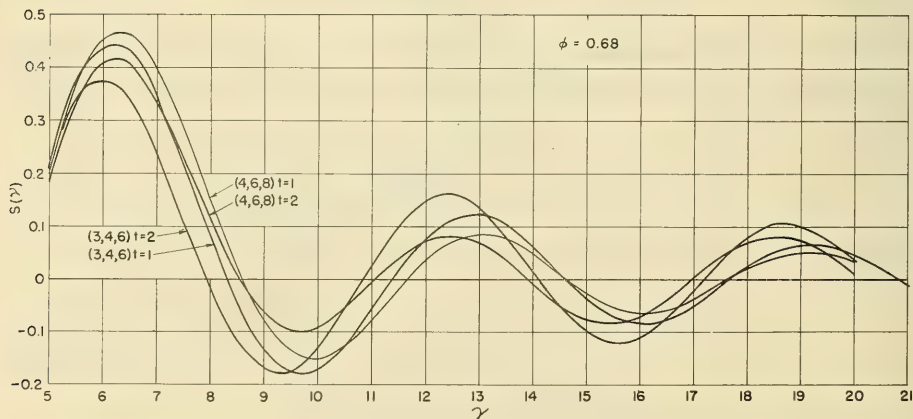


Figure 28a

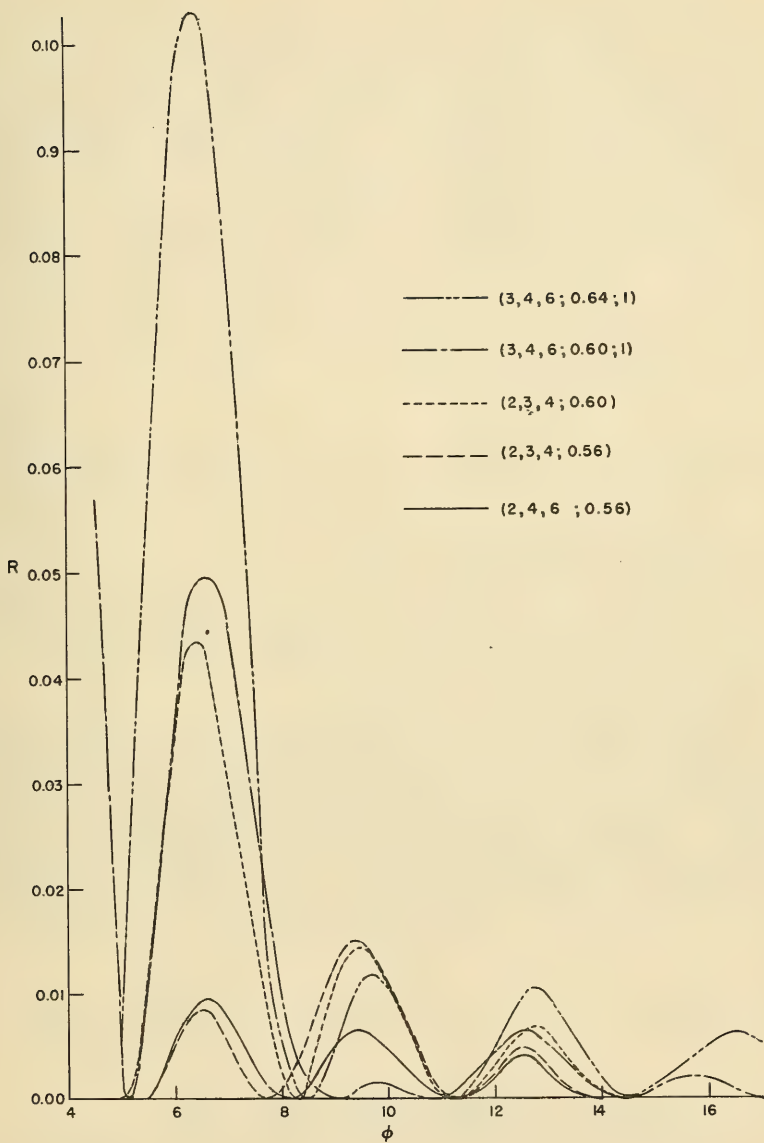


Figure 28b

Figure 28 - $S(\gamma)$ for Some Lines $\phi = 0.68$

equation of the actual water line of the plate or at least its area coefficient (prismatic) is considered. Within the region of the second hump the wave resistance of a "plate body" with a prismatic $\phi \approx 0.9$ (which is nearer to actual practice than $2/3$) can be 10 times higher than that of a parabolic form $\phi = 2/3$. A special investigation was made into "hollow versus straight lines".¹⁰⁹ From this work it follows that theory tends to overestimate the optimum t values. Finally, two asymptotic laws have been derived:¹¹¹

a. For vanishing Froude numbers, the wave resistance becomes proportional to t^2

$$R \sim t^2 \quad [23]$$

b. For very high Froude numbers

$$R \sim \phi^2 \quad [24]$$

The latter law can be easily derived, also, from an asymptotic formula due to Lamb.²⁶

Although the practical significance of these relations is not very great, two important remarks should be made:

a. The $S(\gamma)$ curves here reproduced give only a rough picture. For actual computations the functions must be extended to higher values of the variable γ ; some tabulated values are published⁹⁸ and more comprehensive tables are available at the Taylor Model Basin.

b. Theory deals only with that part of the hull which is submerged at zero speed. Obviously the hull above the load water line at rest must have some influence on the resistance. This point is strongly supported by Eggers' experiments.⁵⁶ However, resistance measurements made on models with different section forms above the load water line (see Figure 29) did not disclose appreciable differences in resistance results.⁹⁸ Of course two experiments cannot disprove the consistency of the preceding reasoning, and much remains to be done on the subject following an outline given by Guilloton.

5.4. THE INFLUENCE OF THE VERTICAL DISTRIBUTION OF DISPLACEMENT ON THE WAVE RESISTANCE

5.4.1. The Influence of the Midship Section Coefficient

The problem may be stated as follows: In the equation $\eta = X(\xi) Z(\xi)$, the function $X(\xi)$ is kept the same; variations in resistance have to be investigated for various midship sections defined by the function $Z(\xi)$.

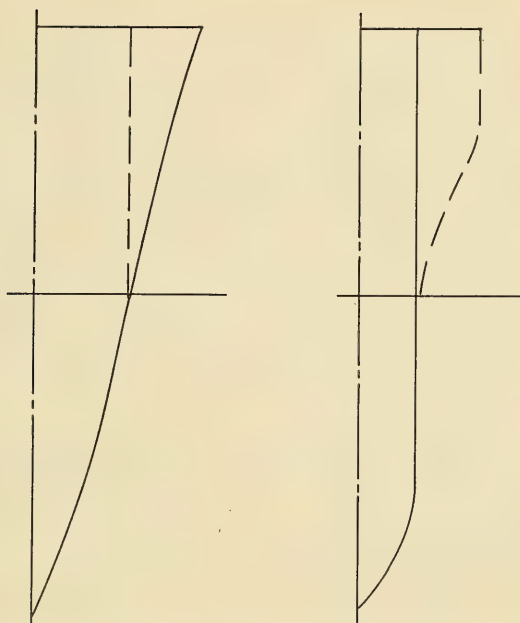


Figure 29 - Changes in Bow Sections Above the D.W.L.

Within the range of normal midship section forms, similar to parabolas of higher degree, $Z(\xi) = 1 - \xi^n$, it is found that the wave resistance is not sensitive to changes of pure shape of section when the area is kept constant. Thus, contrary to the conditions valid for water lines, the resistance qualities of the midship section can be determined in normal cases by the single area coefficient β . Under more general assumptions, the following asymptotic laws are valid:

- a. When F becomes infinite;

$$R \sim \beta^2 \quad [25]$$

- b. When F tends towards zero,

$$R \text{ is independent of } \beta \quad [26]$$

For finite F , R depends upon β , on the ratio $\frac{H}{L} \frac{1}{F^2}$ and on the function $X(\xi)$. Thus no simple rule for calculating R as function of β can be given. Neglecting, however, the dependence upon $X(\xi)$, as in the case of draft, interpolation formulas of the type

$$R \sim \beta^p(F) \quad [27]$$

can be deduced. For normal midship sections, the relations between resistance R and β are similar to the corresponding ones between resistance and draft. Unfortunately, no numerical results of calculations are available at present and no experiments are known which correspond closely to this problem; therefore, we must confine ourselves to qualitative estimates which will be replaced by more adequate data as soon as the pertaining calculations are completed.

In the region of the first hump, the exponent p in Equation [27] is nearer to 2 than to one; hence, small midship coefficients are used in actual design. In the neighborhood of the second hump, high β values are common; this is reasonable although the influence of the midship coefficient on resistance is not negligible, since the use of a large β value admits of the choice of low prismatics without reducing unduly the block coefficient δ .

D.W. Taylor's famous experiments dealing with the resistance R as a function of the midship-section shape⁴² refer to a rather complicated case. They prove, however, that for high Froude numbers the influence of the ratios B/L , H/L and β upon R is similar; for moderate and low F -values, the effect of the ratio B/L becomes predominant. More definite information will follow from the systematic resistance evaluations mentioned earlier.

When the shape of the midship section departs from the normal, special investigations become necessary. These can be performed comparatively easily, however.

In many cases a rule of thumb method proves useful: By shifting some of the submerged volume near the surface vertically downwards the resistance is decreased. The effectiveness of such a change increases with the ratio of the distance d by which the volume is moved to the length of the free wave.

$$\frac{d}{\lambda} = \frac{d}{2\pi F^2 L} \quad [28]$$

i.e., decreases with increasing Froude numbers.

5.4.2 Shape of Sections, Load Water Line and Sectional-Area Curve

Let the sectional area curve $A^+(\xi)$ be given.

The problem of developing good sections can then be replaced to some extent by the simpler one of developing a suitable load water line $X(\xi)$, at least from the point of view of minimum wave resistance.

Within the range of validity of the present theory it can be inferred that reasonable changes in the shape of the sections have only a secondary influence on the wave resistance once the $A^+(\xi)$ and $X(\xi)$ are fixed.

Thus, comparatively simple types of surface equations appear applicable (see Appendix 1, Equation [41]).

As a starting point, the fact is used that parts of the displacement near the surface offer more resistance than those near the bottom of the ship. Hence, U-shaped sections appear to be superior to V-shaped sections as far as wave resistance is concerned. Since for moderate Froude numbers the forebody produces more waves than the afterbody, R.E. Froude's rule* for design can be deduced: U-shaped sections for the forebody, V-shaped sections for the afterbody (the latter with the purpose of reducing "eddy resistance").^{98,99}

Actual calculations prove the superiority of U-forms in almost all cases; however, under exceptional conditions a V-shaped forebody may be superior. Experimental results of this kind were found by Lindblad²⁸ for a fast cargo ship form.

A physical explanation can be given, which will be applied later to bulb forms: Although the wave resistance generally decreases by increasing the depth of immersion of displacement, favorable interference effects may be reduced by shifting some parts of the volume vertically downwards.

In general, agreement has been found between theory and facts, although important examples are known where V-shaped forms were superior to U-shaped forms, contrary to theoretical deductions.^{44,107} The following effect has been stated experimentally for Froude numbers $F > 0.33$: V-shaped models with low or moderate t -values prove to be better than U-shaped models with high t -values although both features (U-shape and large t) are advantageous in the light of theory. In such cases the resistance qualities cannot be predicted from the sectional-area curve alone; Taylor's well known diagrams representing R/Δ as functions of t and $v/\sqrt{L}$ are no longer applicable. No theoretical explanation of these anomalies has so far been found, so presumably the attitude of the model has an important bearing on the subject. When hull shapes depart appreciably from the double wedge form, as for instance in the afterbodies of destroyers, we can no longer rely on the results of resistance calculations, especially in the range of high speeds, without introducing more consistent physical concepts of the phenomena. Consequently, one must be cautious in applying the simple rule of thumb given for the influence of vertical displacement distribution in complicated cases.

*Obviously, considerations of propulsion can change this rule!

5.4.3 Bulbs and Cruiser Sterns

Actually the sectional-area curves of most present day ships are characterized by a condition $t \approx 0$, due to the more or less heavy rake of the stem. As long as the rake is small we suppose that its influence on the resistance is not large. In such cases, we therefore determine t as though the normal form extended up to the forward perpendicular; the actual rounding off at the stem also tends to compensate the error committed. The resistance calculations can thus be performed in the usual way.

Matters become more complicated for heavily raked stems like the well known Maier form. Here the generating singularities should be distributed within a triangular contour at the bow. Although in principle, calculations of wave resistance do not present difficulties even in this case they become rather tedious, so that the properties of such distributions have not been rigorously investigated. Actually one substitutes a distribution limited by a rectangle, keeping the intensities approximately equal (Figure 30). Eventually a more elaborate calculation will be made instead of this rather coarse

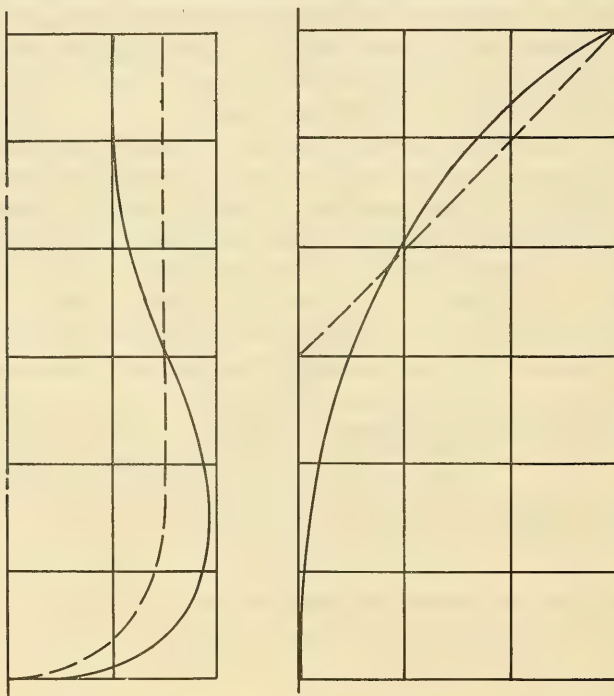


Figure 30 - Pronounced V-sections at the Bow

procedure, although we cannot expect that Michell's theory will give a complete answer for forms with a steep inclination of the sections at the bow.

Still more serious objections can be made when applying the theory to hulls with bulbous bows; but nevertheless the resistance properties of bulbs have been studied with success.

The "bulb effect" is a wave-making phenomenon. In principle the pearlike shape of the bow sections is not a necessary attribute of a bulb form; the latter is defined by the shape of the water lines or of the sectional-area curve, for instance by the ratio f proposed by Taylor (Figure 30a).

However, the pearlike form has resulted from the necessity of avoiding spray formations, which arise when the load water line is rounded off by a large radius.

Replacing the bulb by a sphere (doublet) located at the bow (and later at different distances from the bow) Wigley explained very clearly how the wave trough which generally starts just abaft the sphere diminishes the bow wave of the normal ship and thus also the resistance. He further succeeded in demonstrating that the most advantageous position for the bulb was, generally speaking, just at the bow over the whole useful range of speeds.⁹¹

For practical work it is preferable to express the bulb by a high-power polynomial;¹⁰⁵ the resistance effects of this bulb and of any normal form can be easily combined using the appropriate $S(\gamma)$ functions. Figure 25 indicates how the $S(\gamma)$ function corresponding to a normal form is favorably influenced by a bulb of definite shape and strength. The bulb shape is fixed by the equation of the polynomial used; the strength of the bulb is denoted by a constant factor "a" by which the polynomial is multiplied. Obviously for a given hull, bulb shape and Froude number, the strength "a" will have an optimum value, which theory seems to overestimate.

By the method proposed, we get a much closer description of the bulb form than by Taylor's rather summary procedure:

a. It is easy to show that the efficiency of a bulb depends both on its own shape and upon the character of the ship lines. Generally speaking, the bulb is more advantageous for hollow than for straight lines.

b. The advantage of the bulb disappears at low speed-length ratios; its lower limit of effectiveness depends on the shape of the normal ship form. For hollow lines it is about $F \approx 0.2$ or even less, while for straight lines it may be as high as $F \approx 0.24$ or 0.26 . Theory indicates as upper limit for the application of a bulb a Froude number of approximately 0.6, the exact value depending somewhat upon the form of the bulb and the ship; in fact the limit is somewhat lower.

c. Generally a suitable bulb improves bad forms by a greater absolute amount than it does good ones, except in cases where the inefficiency of the form is caused by features like an exaggerated t value. Reference is made to Figure 15; there the good model 1370 could not be improved by a bulb, which, applied to the bad model 1337, proved to be quite effective. Theory suggests the use of a bulb when rather full forms are driven beyond their limit of economical speed.

d. Generally a bulb uniformly distributed over the draft influences the wave resistance of a normal form more than a submerged bulb (Figure 30a) provided the sectional-area curve of both bulbs is the same. This can be easily understood by the fact that wave effects, hence interference effects too, are stronger at the surface than near the bottom of the ship. In Wigley's paper⁹¹ a different statement is made; however, it seems to be due to the rather abstract shape of the bulb used. Since the uniformly distributed bulb cannot be used because of the spray formation, the submerged bulb is the only practical solution. Probably the loss of efficiency mentioned above can be compensated by a higher strength factor "a".

e. In an ideal fluid the optimum solutions are symmetrical with respect to the midship ratios, i.e., bulbs should be fitted both at the bow and at the stern. Obviously this deduction may be wrong for viscous flow. It has been tried¹⁰⁵ on different models with the result that: 1) For higher Froude numbers, say $F \geq 0.3$ the symmetrical combination had nearly double the effect of a bow bulb alone, and 2) a model with a stern bulb alone did not show any advantages compared with the normal form. The idea of a stern bulb may have some merits with respect to the interaction between propeller and ship.

In summary, the application of theory to bulb forms means a serious violation of the assumptions on which Michell's integral is based. Nevertheless results are obtained which are useful for guidance in research and design work.

As for the cruiser stern, by fitting this type of afterbody the total resistance of a model may be reduced at some Froude numbers while, for other ones, the success may be negligible.⁴⁷ This dependence upon speed indicates that at least one part of the beneficial effect is due to wave interference.

The phenomena involved can be investigated using a method proposed by Havelock for calculating the influence of viscosity on wave resistance (page 63). Numerical computations are under way. Similar calculations can be made when dealing with a possible influence of bossings on the wave resistance.

5.4.4 Ships of Least Resistance

Ship-resistance research will have reached its practical goal when we will be able to indicate the form of least total resistance for any given conditions of speed, displacement, etc. Theory emphasizes the well-known fact, often forgotten by inventors, that there does not exist one ship form of least resistance, but that optimum forms vary with Froude numbers and with other conditions. So called optimum shapes like "pisciform" (fish form) which have been derived from considerations valid for an unbounded fluid lack any serious background for surface ships.

The problem of finding ships of least total resistance can be formulated analytically; however, this formulation does not seem to be helpful as long as no analytical expression for the viscous drag is known. The frictional resistance may be assumed with reasonable accuracy to be proportional to the wetted surface.

Thus, the problem of calculating ships of least wave resistance appears to be the appropriate first step towards the solution of the more general task. In fact, the wave resistance is the "component" most sensitive to changes in form and is responsible for the dependence of optimum form upon Froude number. We can hope, therefore, to obtain the most essential information on ships of least resistance by solving the problem for the wave resistance—an assumption which underlies Froude's method.

From the form of the resistance integrals for ships, submerged bodies of revolution and pressure systems, it follows that the forms of least resistance are symmetrical with respect to the midsection, since then the term I^2 (see Appendix 2, Equation [12]) becomes zero.

Another deduction which will be needed later is that the resistance of an asymmetric body is the same when moving ahead or astern. These results, which are contrary to our general experience, are caused by the assumption of an ideal medium; at high Froude numbers, however, the effect of viscosity is small, so that for a restricted class of bodies symmetry may become an approximate condition of least wave resistance as has been shown by experiments. Actual ship forms suitable for very high speeds, however, do not comply with the condition of symmetry, one reason for the departure from the results of simplified hydrodynamic theory being the influence of the changed attitude of the ship at such speeds (trim and bodily rise).

Keeping in mind that the following results must be applied with caution to actual conditions, we discuss methods used, the difficulties met with, and the practical information obtained when trying to find ships of least wave-making resistance.

To eliminate the trivial answer that the wave resistance vanishes for vanishing displacement some additional "restraint" must be introduced; the most important condition is to assume a constant displacement. Also, the conditions of a fixed midship section $A_M = \beta \times B \times H = \text{const}$, and of least specific resistance are of theoretical interest. Thus, a number of isoperimetric problems in wave resistance are formulated.

From an inspection of Michell's integral as well as from physical reasoning, it is obvious that a rather trivial answer exists as to the best vertical distribution of the displacement. Since the influence of the wave making decreases with $e^{-\frac{2\pi z}{\lambda}}$ the displacement should be concentrated as far below the water line as possible or the draft should be infinite. Thus, even when the volume is fixed, additional restrictions on the draft and the shape of the transverse sections are necessary. The essential remaining problem to be solved is the optimum longitudinal distribution of displacement.

A plausible simplification is to substitute an infinite draft as long as only general information is desired; but such an approach is not suitable when detailed practical results are needed.

Some results have been found by applying Ritz's method which, however, are valid only for the restricted kind of functions used (Figure 31). To quote from an earlier paper¹⁰¹ "As a further difficulty it may be mentioned that the assumption of the type of surface equation involves a highly arbitrary element, and very advantageous forms can remain outside the scope of our considerations by lack of knowledge of their analytical representation;"

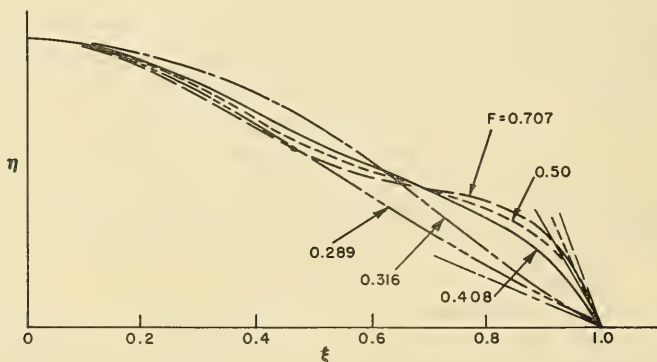


Figure 31 - Sectional Area Curves for Ships of Least Resistance

"Considering all the circumstances" (the approximate character of the hydromechanic theory and the choice of the hull equation in order to use Ritz's method) "one should not expect to obtain a final solution of such a difficult problem by heaping up approximations..."

Later it was shown by Pavlenko,⁴⁰ von Kármán²³ and Sretensky³⁷ that the solution of the problem was hampered by serious mathematical difficulties. All three authors used infinite draft for their final deductions. The condition $\nabla = \text{const}$ reduces, in the case of infinite draft, to $A_W = \text{const}$, i.e., the final answer represents the best shape of the water line, which can also be interpreted as the shape of the sectional-area curve.

Following von Kármán, an exact solution of the problem of calculus of variations exists only for a limited range of medium Froude numbers. This statement agrees to some extent with Pavlenko's analysis. L. Sretensky, however, denies the existence of any solution by square integrable functions over the whole speed range. Because of the fundamental theoretical importance of the problem, at present Wehausen of the Taylor Model Basin is reconsidering the matter.

The naval architect's point of view is somewhat different from that of the mathematician's. Michell's integral is only an approximate solution even in the case of an ideal fluid, therefore we must check experimentally any optimum form derived from it. The physical meaning of such results is decisive; exact solutions of the integral (if they exist) may be less valuable than approximations which yield results within the important range of the theory.

The practical results reached may be summarized as follows (see Figures 31 and 32):

The optimum longitudinal and (within restricted limits) vertical distribution of displacement agree well with experimental work, the latter due mainly to Taylor.⁴² In particular some optimum values of $\phi = C_D$, t , $\beta = C_M$ found experimentally were in agreement with theory. "Swanneck" forms were rediscovered and some new features like "small swannecks" (indication of a bulb) found when studying moderately full vessels. Only restricted use can be made of Pavlenko's forms (Figure 32); the extremely blunt sectional-area curves (or water lines) probably indicate that the theory has been overstressed. On the whole, theory has lagged behind experiment—partly because of the difficulties in principle, partly because of the inadequate effort applied to the subject. The methods of computation have hitherto admitted the use of only two arbitrary parameters, and more widely applicable approximate results can be expected by improving the methods of calculation.

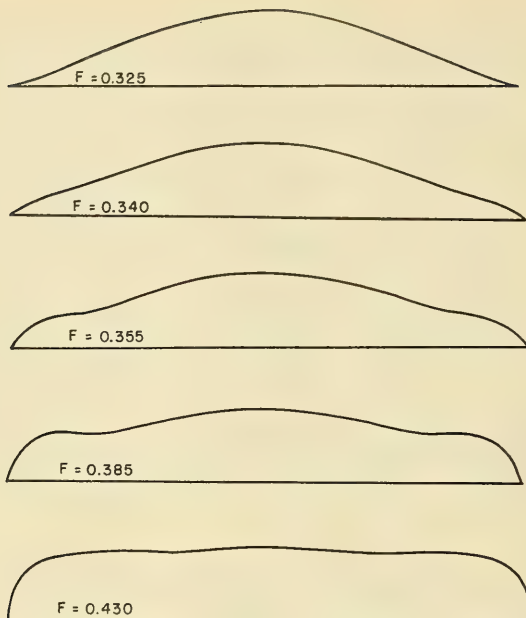


Figure 32 - Sectional-Area Curves for Ships of Least Resistance and Infinite Draft (Pavlenko)

A similar investigation on wholly submerged bodies of revolution has proved rather fruitful, particularly because of a lack of any earlier work on this subject.¹¹¹

Optimum sectional-area curves (optimum doublet distributions) are similar to corresponding curves of surface ships, but peculiar form effects seem to be more pronounced with submerged forms.

Finally, we must return to our point of departure—ships of least total resistance. Adding a frictional-resistance term to the wave resistance

$$R_t \approx R_w + R_f \quad [29]$$

where R_f is assumed proportional to the wetted surfaces, and applying Ritz's method to R_t , a closer approximation can be obtained. Within the range of examples investigated, no great differences were found between the forms calculated from conditions of minimum wave and minimum total resistance as given by [29]. However, it is expected that such a procedure may eliminate some

mathematical difficulties. A further step can be made by using some corrections for viscosity, as shown later, but taking into consideration the somewhat arbitrary character of these corrections no results of general value can be achieved.

6. INFLUENCE OF VISCOSITY

So far emphasis has been put on the wave resistance of symmetrical ships, since in an ideal fluid the even terms of the surface equation contribute the principal part of the resistance as long as normal forms are considered. The resistance "component" corresponding to the antisymmetrical part is computed in the same way as for the main part by slightly different functions; the two components can be simply superposed. As mentioned before, in an ideal fluid the wave resistance of asymmetrical bodies is the same when moving either ahead or astern.

Obviously, this paradox does not hold for a real liquid. When asymmetrical variations of the surface are made, resistance calculations valid for an ideal fluid in many cases do not agree with the facts even as to sign. Thus, the classical problem of determining the optimum longitudinal position of the center of buoyancy involves the consideration of viscous effects.

Because of our restricted theoretical knowledge, we have to rely upon experimental data; but attempts have been made to deduce from such data more general results. The phenomenological approach used is based mainly on measuring the resistance of asymmetrical models run in both directions. The most important experiments are due to Wigley.

Some practical deductions can be obtained from some earlier work at the Berlin Tank.¹⁰⁸ By using four symmetrical models appertaining to the family (2,4,6; ϕ ; 1) with $\phi = 0.52, 0.56, 0.60, 0.64$ and two asymmetrical models $\phi = 0.56, 0.60$ run in both directions ($\phi_F = \phi = 0.04$), the following deductions were obtained:

a. For the fuller model $\phi = 0.60$ the shifting of the center of buoyancy aft proved to be advantageous except for $F > 0.4$.

b. For the finer model $\phi = 0.56$ the aftward movement of the LCB gave a model worse than the symmetrical basic model.

c. Thus the influence on the resistance of the shifting of the LCB for given Froude numbers depends on the form (equation) of the original surface and the method of moving the LCB (equation of odd members added). As a rule resistance properties of asymmetrical ships cannot be described by the position of LCB (or the difference $\phi_F - \phi_A$) alone.

d. For moderate F the resistance properties of the forebodies (judged by the resistance qualities of symmetrical models) yield an estimate of the success obtained by moving the CB.

Conditions for forms with higher prismatics are especially complicated, since here the viscous-pressure drag may become as sensitive to form variations as the wave resistance. These two resistance components influence the total resistance in opposite ways. For low wave making we need a fine forebody; for low viscous-pressure drag a fine run. The LCB of full slow ships lies normally forward of the midsection, although the wave resistance is increased by this location, since in this case, the viscous-pressure drag is more important than the wave resistance.

An additional complication is caused by a possible instability of flow, to which this kind of model is especially subject. Thus, the analysis is peculiarly unsatisfactory when dealing with the important class of slow and moderately fast cargo ships, because:

- a. The theory of wave resistance can be applied only with great restrictions on account of their full forms and low Froude numbers.
- b. Our knowledge of viscous-pressure resistance and frictional-form drag is completely inadequate.
- c. Numerous earlier experiments are questionable as to their accuracy.

Various attempts have been made by Havelock to estimate the influence of viscosity on wave phenomena. In the first place, Havelock pictured the process phenomenologically in terms of a friction belt whose effect may be equivalent to reducing the slope of the ship towards the stern; the equivalent shape of the hypothetical body was rather arbitrary.⁶² The corresponding resistance curve showed much less waviness; the greatest part of this effect can be explained by the heavy reduction of the prismatic due to the virtual lengthening of the form.

The next step consisted in introducing a correction factor β_v^* to allow for a decrease in efficiency of the elements of the ship's surface in going from bow to stern;⁷¹ the frictional effect is treated as a diminution in the effective relative velocity of the model and the surrounding water. This method was developed by Wigley, who applied the same factor β_v^* for the decay of the bow waves.⁹⁴ He deduced the reduction factor β_v^* from resistance curves by finding the values necessary to give reasonable agreement between theory and experiment. Important practical results thus obtained are discussed later.

From a theoretical point of view, the procedure leaves much to be desired. The factor β_v^* is assumed to be constant over the whole after half of the ship, while the influence of the viscous phenomena is undoubtedly concentrated at the extreme end of the run, where the generation of waves is most heavily rescued by its effect. Following Havelock, this fact alone explains why the influence of viscosity on the wave pattern is so much more pronounced for low Froude numbers.⁷⁷

Havelock calls a third most promising step "an illustration of the possible effect of boundary layer on wave resistance;" Small modifications of the lines near the stern are made so as to obtain the required kind of change in the calculated resistance curve.⁷⁷

The displacement thickness* of the frictional layer (something of the order of one-tenth of the boundary-layer thickness) is inappreciable, except at the stern of the ship, where, because of the reduced girth and eventual separation, a wider wake is created.

Quantitative estimates of viscous effects on wave patterns appear to be possible when an appropriate singularity distribution is found which takes into consideration the influence of viscosity.

With the kind permission of the Institution of Naval Architects, two figures (Figures 33 and 34) are reproduced from a paper by Wigley, which summarize some important resistance results. The hulls investigated belong to a family (2,4,6); asymmetry is produced by adding a term $a_3(\xi^3 - \xi^5)$.

The investigation is similar in purpose to the corresponding one in the first part of this chapter, but more elaborate computations have been performed. Instead of the resistance curves, differences in the resistance between appropriate models are plotted and experimental values are compared with results of calculations with and without viscosity correction. In Figure 33 the symmetrical basic Teddington Model 1970B sectional-area curve (2,4,6; 0.7; 2) is compared with an asymmetric model 2170A derived from 1970B by shifting the center of buoyancy by 0.02L.

When the full end of 2130A is leading, a reasonable agreement is obtained between measured and calculated resistance without any viscosity correction. However, when the fine end is leading the concept of ideal fluid breaks down, while the semiempirical viscosity correction yields at least a qualitative agreement.

*The displacement thickness δ^* is the amount of displacement by which the main stream is thrust away from the body due to the slowing down in the boundary layer. The mathematical expression is

$$\delta^* = \int (1 - \frac{U}{U_0}) dy$$

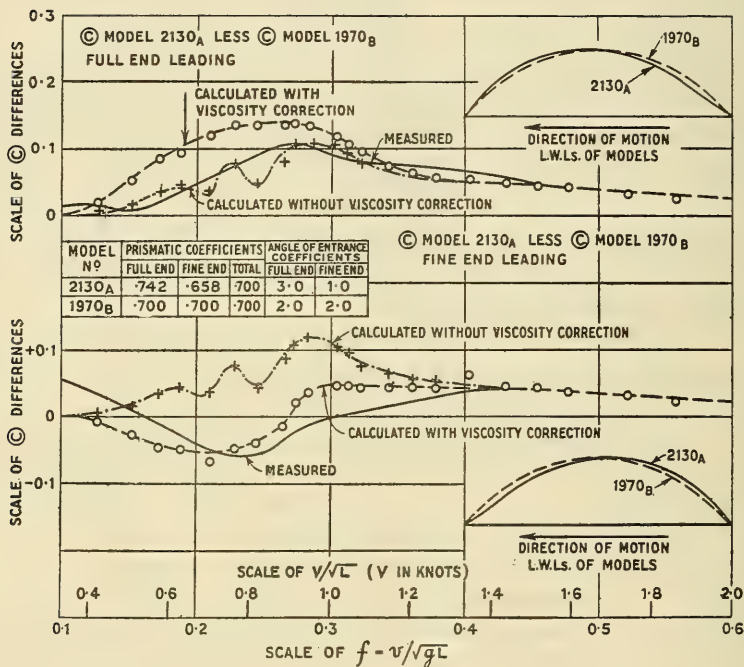


Figure 33 - Effect of Asymmetry on Resistance (Wigley)

(Courtesy of Institution of Naval Architects)

The effect on resistance of a reversal in the direction of motion is given by Figure 34; only calculated curves with viscosity correction can be compared as without correction the difference is obviously zero.

Interesting results have been obtained for the influence on resistance of changes in the forebody only, the afterbody being kept constant, and vice versa.

For variable bow shape, calculations without viscosity correction are in good agreement, on the average, with experimental data within the range of forms investigated.

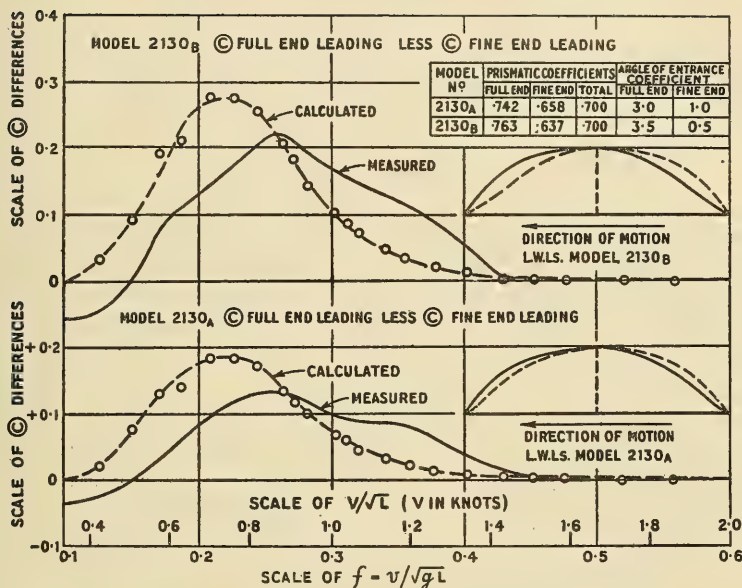


Figure 34 - Effect on Resistance of Reversal of Direction
(Courtesy of Institution of Naval Architects)

For variable stern shape, the results depend upon the type of hull. By fining the stern of a full symmetrical vessel, only negligible or moderate reductions in resistance were obtained up to a Froude number of $F = 0.3$ (thus "Froude's rule" proved to be consistent), while by increasing the after-body displacement of a finer ship a more pronounced increase was found.

7. WAVE PHENOMENA DUE TO THE PROPELLER ACTION

The interaction between ship and propeller can be successfully studied by the method of singularities. In particular, the mutual forces between hull and propeller can be calculated by Lagally's theorem as soon as appropriate images are known. Source and sink distributions which picture the hydrodynamic properties of a ship have been already discussed. As far as the

problem of interaction is concerned, a suitable "model" for the propeller is furnished by a sink distribution over its disc and for greater distances even by a single sink only. The "input" is given by

$$Q = Av\theta \quad [30]$$

where θ is the ratio of the slip stream to the propeller speed, and A the propeller disc area. Denoting the flux through the propeller disc by Q , the thrust is written

$$T = \rho Qv \left(1 + \frac{\theta}{2}\right) \quad [31]$$

From Lagally's formula a force of attraction is obtained between the hull and the working propeller, since the afterbody represents a sink system which is closer to the propeller sink than the positive forebody system. The internal force is the thrust deduction or resistance augmentation in an ideal fluid. One can further compute the wave resistance due to the working propeller alone. Dickmann found that for normal conditions of depth of immersion and loading factor this resistance is a small fraction of the thrust, say of the order of 1 percent; it depends on a Froude number $v/\sqrt{gh}$ (h immersion) and can reach greater values for high loading factors and small immersions. The interference between ship and propeller waves causes more important effects.

Following F. Horn¹¹⁵ the propulsive efficiency depends on the position of the propeller relative to the wave created by the hull; in a wave crest the orbital motion is directed horizontally forward, thus generating a favorable positive wake, while in a trough conditions are just the opposite. Dickmann^{9,10,11} has proved Horn's reasoning in a rigorous manner by means of wave theory; he has further shown that beautiful experiments by Yamagata support his conclusions. In the light of these ideas wave patterns at the location of the propellers should be more carefully studied.

Thus, the theory developed by Dickmann has succeeded in revealing the mystery which for a long time enveloped the intricate problem of interaction between hull and propeller; but one is far from a complete quantitative solution. We may note some facts where no complete agreement between theory and experiment has been reached.

The thrust deduction (contrary to the wake) should be fairly independent of wave phenomena or, otherwise expressed, the part of the thrust deduction coefficient due to waves t_w is very small. Some experiments, however, indicate that the thrust deduction can be influenced by variations in the entrance of a hull which to our knowledge affect measurably only the wave

pattern.¹¹⁴ This discrepancy in the behavior of t_w introduces an element of uncertainty in our reasoning which should be eliminated as far as possible by reliable experiments. Another disturbing fact is the bad quantitative agreement between the values of the suction force derived from the difference between thrust and resistance and those evaluated from pressure measurements; this discrepancy is not due primarily to wave phenomena.

The fact that the wave formation around a hull is only slightly influenced by the propeller action gives a valuable support to our present technique of model towing phenomena; a model test without screws discloses a basic property of the hull which normally remains unchanged in a self-propelled condition. Additional suggestions are made by theory:

a. To locate the propeller in a region of a high wave wake.

b. To change the wave pattern of the hull itself in such a way that a high wake may be created at the propeller disc. This viewpoint has not hitherto been applied in model research; it is even doubtful if it would lead to practical results.

Dickmann has extended his analysis to actual flow. He has explained why non-uniformity of the wake increases the thrust deduction.*

In the light of this knowledge and our remarks on stern bulbs an idea due to E. Hogner and G. Kempf¹¹⁶ may be reconsidered. These authors proposed to create an axially symmetrical wake by giving an appropriate shape to the run. Unfortunately, according to van Lammeren, gains in propulsive efficiency so reached are counterbalanced by increased resistance.

However, it is worth while investigating whether with such a run a stern bulb effect can be obtained which combined with a bow bulb may reduce the wave resistance. The remark applies to high-speed single-screw ships as well as for triple-screw vessels.

Finally the possible influence of propeller suction on separation over full sterns has been often mentioned. Unfortunately this idea cannot be dealt with in a reliable way until adequate research is carried out into the flow patterns at the stern.

8. RESISTANCE IN RESTRICTED WATER

Two types of problems are met with: Motion in shallow water, in which only the depth is limited, and motion in canals.

*van Lammeren has deduced this statement from experiments.³⁴

Only a short enumeration of the most important facts is possible. The wave pattern of a ship advancing with a constant speed v in shallow water differs from the corresponding pattern in infinite depth by an increase in the wave length λ and by a change in the configuration of the waves, more energy being stored in the echo waves.

The existence of a critical velocity is due to the fact that for $v > \sqrt{gh}$ the transverse waves must disappear.

The phenomena are complicated, but useful results have been reached by a simple method devised by Schlichting⁵² which yields an estimate of the resistance in shallow water when a resistance curve for infinite depth is given. Schlichting introduces the hypothesis that the resistance in deep and shallow water is the same when the length of the free wave corresponding to the ship speed is the same. Thus differences in height and configuration of the wave pattern are neglected. Using well known formulas for the wave (ship) velocity

$$v_0 = \sqrt{g\frac{\lambda}{2\pi}} \text{ (deep water) and } v_h = \sqrt{g\frac{\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}} \quad [32]$$

(shoal water) it is easy to calculate the shallow-water speed v_h from the condition $\lambda = \text{const}$, when v_0 and h are given. Experiments give considerable support to this rather bold idea, but the method fails at speeds above the critical.

A hydrodynamic solution for shallow-water resistance has been given by Sretensky;³⁸ it is valid approximately under the same assumptions as Michell's formula (see Appendix 2). Sretensky's integral has been used to demonstrate that Schlichting's hypothesis has some theoretical foundation in that, within the subcritical range, the most important phenomena can be deduced from it.¹⁰

We summarize briefly some important points:

a. An additional Froude number $F_h = \frac{v}{\sqrt{gh}}$ is useful; $F_h = 1$ for $v_c = \sqrt{gh}$, the critical wave speed.

b. Shallow water effects become appreciable only when $F_h \geq 0.7$; generally below this limit the water can be considered as infinitely deep (in so far as wave resistance is concerned).

c. Obviously the common Froude number $F = \frac{v}{\sqrt{gL}}$ is connected with F_h by the depth-length ratio

$$\sqrt{\frac{h}{L}} = \frac{F}{F_h}$$

For general analysis of resistance properties in shoal water, the ratio h/L is more characteristic than the draft-depth ratio H/h . As usual, Taylor's representation of the matter⁴¹ based on F and h/L is superior to others.

Within the validity of the theory (H/h small) H/h is a parameter whose changes cause only moderate deviations from corresponding deep water phenomena when h/L is kept constant. Hence, it is in principle disadvantageous to link up wave effects with the draft H instead of the length L .

d. Besides the critical speed defined by $v_c = \sqrt{gh}$, another critical speed v_m (the speed of maximum resistance) is sometimes used; we prefer to restrict the terminology "critical speed" to $v_c = \sqrt{gh}$.

The maximum wave resistance occurs at a speed $v_m < \sqrt{gh}$, the difference $\sqrt{gh} - v_m$ increasing with increasing h/L .

e. The maximum wave resistance in shoal water for a given displacement does not seem to depend much on the ship form. This can be inferred (within the validity of the theory) as follows: The speed of maximum resistance in shoal water corresponds to an extremely high speed in deep water; but in the latter case, displacement is decisive and the resistance is nearly independent of form.

f. When h/L becomes large, say of the order of unity, so that F_h critical corresponds to $F \approx 1$, the difference between deep and shallow water resistance is not too large. This is the reason why resistance tests with models of high-speed ships yield useful results for deep water, although they may run in the range of the critical speed.

The motion of a ship in a canal can be treated by the methods of engineering hydraulics. A clever application was made by Kreitner;¹² he explained supercritical conditions by analogy with the hydraulic jump and gave information on the average speed of flow of the water around a ship in a canal. The study of motions in a canal, besides its immediate practical application to canal and river shipping, also has a bearing on foundations of model testing.

In a canal the speed for maximum resistance occurs somewhat earlier than in shoal water of unlimited breadth. O. Mueller tries to explain this fact by introducing an empirical Froude number⁵³

$$B = \frac{v}{\sqrt{gh}} \sqrt{2\frac{h}{b} + 1} = F_h \sqrt{2\frac{h}{b} + 1} \quad [33]$$

substituting the hydraulic radius $\frac{bh}{2h+b} = \frac{h}{\frac{2h}{b} + 1}$ for the depth h .

Theoretical solutions of the problem are due to Keldysh and Sedov⁵⁰ and Sretensky.³⁹ Using the latter's formula, some estimates were made of the effect upon the wave resistance of different size models with a parabolic waterline in rectangular tanks.¹⁰⁰ When properly extended, such evaluations should lead to correction factors for the influence of the limited cross section. Obviously the task is much more complicated than the corresponding one in wind tunnels. To cover adequately the whole field we need results for:

- a. The whole useful range of Froude numbers.
- b. Different ratios h/b .
- c. Different ratios L/b .
- d. Different characteristic ship forms.

Preliminary calculations indicate that we have to distinguish roughly between deep- and shallow-water conditions (say $h/b \approx 0.5$ and $h/b \leq 0.1$). For deep-water tanks we distinguish between conditions far from the critical speed ($F_h = 1$) and near to it. When $F_h < 0.7$ and $L/b < 0.5$ the limited breadth of tanks does not seem to influence the wave resistance appreciably, although a recent publication by Wigley does not support even this assumption.⁹⁷

A wave resistance increase up to 15 percent has been found at high Froude numbers when using a model length ratio $L/b = 1$ instead of $L/b = 0.5$. Generally, an increase in resistance due to tank walls is found, when the resistance curve for infinite liquid has a tendency to rise. At some Froude numbers the resistance in the canal is smaller than in open water. Models with a length $L = b$ are liable to furnish totally "wrong" resistance results at some Froude numbers when applied to deep-water conditions.

When the intended correction factors will be available we shall be able to indicate upper limits of model sizes as given by wave phenomena, while lower limits are fixed by conditions of viscous flow.

Somewhat surprising results are found for shallow-water conditions, $F_h = 0.9$: Keeping the canal breadth b constant but doubling both the size of the model (II) and the water depth ($h/L = \text{const}$) the calculated wave-resistance coefficient of the large model was nearly twice that of the small one, Figure 35. The values were

I	II
$h/b = 0.05$	$h_1/b = 0.1$
$L/b = 0.5$	$L/b = 1$

$h/L = 0.1$ remaining the same in both cases. But even for Arrangement I the resistance coefficient is some 35 percent higher than for the corresponding case where $b \rightarrow \infty$ and $h/b \rightarrow 0$.

Near the critical speed, model results must be converted cautiously into full-size data valid for unlimited breadth. On the other hand, at $F_h = 0.8$ the resistance of the models discussed is practically uninfluenced by the finite breadth b .



Figure 35 - Models in Shallow Water Basins

Having in mind the oscillatory character of wave effects, one cannot expect too much from the Equation [32]; perhaps at the best it can explain the earlier rise of the resistance curve due to the finite cross section near the critical speed. Some experiments made at the Hamburg Tank appear to support the theoretical reasoning.⁵⁴ Comstock and Hancock's paper⁵¹ will furnish valuable checks, although it is thought that the influence of the finite cross section on wave resistance has in principle a more complicated character than represented by the curves on Figure 18 of the paper quoted.

9. EXTENSION OF THEORY*

Not all shapes of bodies of revolution can be represented by singularities distributed over the axis. *A fortiori*, not all ship forms can be generated by sources and sinks distributed over the center plane; even the H/B ratio attainable by this procedure is limited to $H/B > 0.5$. Furthermore, it is impossible to expect that highly curved parts of the hull such as the bilges can be obtained by a plane distribution.

A first attempt to escape from this restriction was the introduction of volume singularities proposed by Hogner.²⁰ His "interpolation formula" agrees in the limit of vanishing draft or vanishing beam with his integral or Michell's integral, respectively.

Havelock has criticized Hogner's attempt from theoretical considerations;⁶⁹ nevertheless, a closer agreement between calculation and experiment can be expected by its use. Some numerical work based on the "interpolation formula" showed results which, although not unfavorable, were not conclusive.⁸⁶ Unfortunately, we do not know what kind of body results from the singularities assumed. Another ingenious proposal was linked by Hogner;²¹ who suggested applying influence lines to resistance research.

*This chapter can be omitted by readers interested only in practical results.

The most general solution consists of a distribution of singularities over the ship surface itself.

For this case a resistance integral has been given by Havelock.⁵⁸ However, he himself adds somewhat resignedly: "it is not likely that it would give any better agreement with experimental results; for the more we depart from the simple narrow ship the more necessary it is to take into account the effect of wave motion upon the distribution of fluid velocity around the ship."

Thus three steps can be listed which may lead to a really comprehensive theory:

1. Determination of the distribution of surface singularities corresponding to a given ship form. A solution by an integral equation has been indicated by Kotchine,⁵⁸ and has been discussed for very low Froude numbers, i.e., actually for a deeply submerged doubled body. Here, however, the most important problem consists of finding the changes of singularities generating a given form with speed (Froude number). No solutions so far are known except an investigation of a submerged cylinder by Havelock.⁷¹ It appears therefore to be appropriate to start with the simplest bodies.

2. The calculation of the real attitude of models due to changes in hydrodynamic-pressure distribution and of the resulting resistance. A general approach by Hamilton's integral is imaginable in principle, but nothing can be said as to the practical value of such an attempt. Probably it is more reasonable trying to find corrections for simple bodies as under 1.

3. The representation of the viscous flow by appropriate systems of singularities as a base of resistance calculations.

The difficulties in dealing with this general approach are very serious. Therefore, some simpler methods of improving Michell's integral have been sought.

Havelock recommended for high-speed vessels a method based on concentrated singularities; as originally applied by him, it means even a simplification of Michell's theory—the substitution of concentrated sources and sinks along the center plane for plane surface singularities.⁸¹ In so far as this method leads only to simplified computations, we are not interested in it from our present viewpoint; but it can be generalized by locating these sources outside of the plane.

Appropriate formulas have been developed by Lunde for this case.³¹ When applied to a model investigated by Wigley, Lunde's calculations showed a somewhat better agreement with experiment than computations based on Michell's integral. It is expected that Lunde's method will be a useful means

of research. Again, the difficulty arises of how to find the shape of the body generated by the assumed concentrated singularities.

A rather general method due to Kotchine⁵⁸ may be mentioned, developed with the idea of obtaining all the force components due to free surface effects. Kotchine's elegant expressions have not yet led to new results for the wave resistance, but they have proved to be useful when calculating vertical forces.

Finally, reference must be made to the work of R. Guilloton. Two new basic ideas may be distinguished in his publications (References 15, 16 and 17):

a. The improvement of resistance calculations over those based on Michell's potential only, mentioned before.

b. The use of resistance calculations based on a second approximation to the velocity potential valid for forms with fuller section (flat bottom).

Both methods of calculation are characterized by the use of wave profiles and of the pressure distribution due to wave motion. An original approach based on finite differences leads to solutions for an arbitrary ship form with finite draft.

Figure 36 indicates how far Guilloton's method is superior to Michell's integration of the pressure distribution from a physical point of view; while the latter is confined to the underwater part of the form only, Guilloton extends it over the proper limits. Objection can be made that Guilloton's procedure is liable to lead to errors of computation, since the result is arrived at by evaluating quantities which are relatively small differences of other quantities, themselves known with only limited accuracy. This question can only be decided by actually performing calculations, and the present author has not to date had the opportunity of carrying out such work. However, Guilloton's method of evaluating wave profiles and the wave pressure exceeds as to rigor all the proposals hitherto made; hence, it must be considered as a fundamental work intended to base the computation of wave resistance on an evaluation of the complete flow pattern.

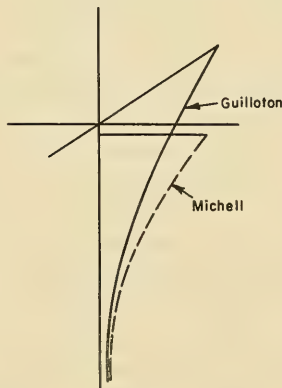


Figure 36 - Pressure Distribution Over a Section due to the Motion of a Ship, Following Michell and Guilloton

10. WHOLLY SUBMERGED BODIES

The wave resistance of wholly submerged bodies deserves a short treatment even within the scope of the present report. We mentioned the problem when treating ships of least resistance and the influence of beam on the resistance of ships with the purpose of elucidating the conditions valid for surface vessels.

More generally, the theory of wave resistance of wholly submerged bodies moving horizontally near to the surface represents an interesting study in hydrodynamics capable of wide applications; it has a fundamental bearing on problems of submarines and torpedoes.

An important problem in tank work is to reproduce conditions for a body moving in an unbounded fluid. In this case we try to establish the minimum depth necessary to avoid wave phenomena or to reduce them so drastically that their influence can be eliminated by rather small correction factors, similar to the well known procedure in aerodynamics. Such considerations were sometimes neglected in earlier experiments and led to doubtful results.

From Michell's (Havelock's) integral a resistance formula can be immediately written down which is valid for a totally submerged system of singularities distributed over the vertical centerplane and which is suitable to generate a submerged body like a submarine, etc., provided the total output is zero. Particularly, distributions symmetrical with respect to a horizontal plane can picture double models which are valuable for resistance research. For a first orientation we may confine ourselves to bodies of revolution, whose image system is given by a line distribution.

The theory is based on the assumption that the depth of immersion f is great compared with the radius b of the resulting body; $f/b \gg 1$. Under this condition the wave resistance of a very elongated body of revolution gives a fair approximation to the resistance of more general bodies of the same length and sectional-area curve, provided the vertical and horizontal maximum dimensions (height H and beam B) do not differ too much. This statement is supported by some calculations made earlier and by a remark due to Havelock⁶⁶ and to some extent by Lamb's formula.²⁶ From a formal point of view the calculation of wave resistance for a body of revolution is simpler than for a surface ship, the same auxiliary functions being used. A matter of primary importance for practical work is to determine the limiting value of the ratio f/b above which the theory may be expected to yield a reasonable result.

Calculations have been checked by experiments for three bodies of revolution over a range of medium Froude numbers (up to 0.42).¹¹⁰ The quantitative agreement was not satisfactory for $F = 0.26$, and some unexpected shift of phase was found between calculated and measured resistance curves. However, a first orientation as to the relative properties of different models can be obtained even for the case $f = b$, i.e., when the backs of the bodies touch the surface. Generally, a closer agreement is found between computations and experiments for submerged than for surface ships as regards the form of humps and hollows in the resistance curve. Some astonishing results were found: For instance due to pronounced interference effects at some Froude numbers > 0.35 the total resistance of a full body ($\phi = C_p = 0.80$) is lower than that of a very fine one ($\phi = C_p = 0.546$) having the same principal dimensions but some 30 percent less volume.¹¹⁰

One must be cautious in applying results obtained from blunt bodies like a sphere or circular cylinder to elongated bodies. For the former the speed $v = \sqrt{gf}$ is a critical value since the resistance curve has a maximum value at that speed (comparable to the case of finite depth h); hence, the use of a Froude number $F_f = v/\sqrt{gf}$ is advisable in plotting results. But resistance curves of bodies having ratios of slenderness comparable to those of ships, $a/b \approx L/\sqrt{A_m}$, do not show any peculiarity at $v = \sqrt{gf}$; therefore the dimensionless number $F_f = v/\sqrt{gf}$ cannot be recommended when investigating the resistance at constant immersions. On the other hand the parameter $v/\sqrt{gf}$ is appropriate when investigating the resistance as a function of the immersion.

Finally, reference may be made to the problem of bodies of least wave resistance. It was mentioned that results are similar to those for surface ships, but peculiarities of form are still more pronounced. For a given Froude number the optimum form varies slightly with the depth of immersion. Figure 37 indicates the shape of some forms derived under rather special conditions. In the light of the remarkable qualitative agreement between calculation and experiments it appears legitimate to develop optimum forms by calculations.

So far no attempt has been made to treat bodies having blunt noses, which cannot be represented by a line distribution.

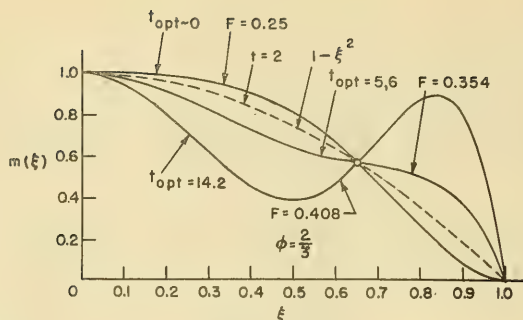


Figure 37a

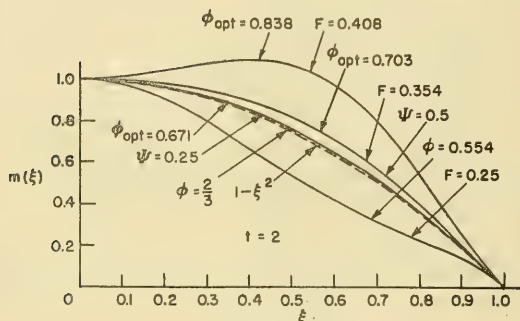


Figure 37b

Figure 37 - Bodies of Revolution of Least Wave Resistance

$m(\xi)$ is the doublet distribution (approximate sectional area curve).

SUMMARY

Pending a thorough tabulation of Michell's integral, the relations between ship form and wave resistance are of necessity discussed here only in a rather broad way.

1. The relations involved are generally complicated. Thus it is not surprising that two seemingly contrary basic properties can be stated:

a. Under certain conditions the wave resistance can be very sensitive to changes in lines; and

b. Widely differing hulls can yield equivalent resistance results for given Froude numbers. Under the headings 2 to 8 below some theoretical results are summarized and under 9 to 16 the results of experimental checks are discussed.

2. From the form of the wave integral it follows immediately that for research the appropriate independent speed variable is Froude number referred to the length of the ship. This rule holds for all slender bodies.

3. When investigating the wave resistance as a function of the ship form, two working hypotheses are widely used:

a. A more basic division, in which the resistance is considered as being a separate function of the principal dimensions (or their ratios) and of the dimensionless shape and,

b. A splitting up of the resistance due to the dimensionless shape into functions dependent upon the longitudinal and vertical distribution of displacement. More general and more rigorous investigations are required to consider the mutual interdependence of most of the factors concerned.

4. The wave resistance R depends upon the square of the beam $R \sim B^2$ and in a more complicated manner upon H . Relations $R = R(H)$ can be established by somewhat cumbersome but not difficult computations. However, the limits of validity of $R(B)$ and $R(H)$ derived from Michell's integral are restricted.

5. The concept of the sectional-area curve, which embodies the longitudinal distribution of displacement, proves to be fruitful both from a theoretical as well as from a practical viewpoint. It has been shown that the basic parameters ϕ and t are indispensable for any research work on the subject; in addition it has been demonstrated that Taylor's curvature parameter κ is valuable when dealing with fine lines. However, even the three parameters ϕ , t and κ may not be sufficient to fix the wave resistance properties of a ship line. For fuller shapes the length of paralleled middle body replaces the curvature κ as the third parameter.

6. From this fact it follows that any systematic research on ship forms should be based on analytically defined lines and surface equations. The most suitable expressions are polynomials.

7. By evaluating the resistance integral for a sufficient number of "longitudinal" polynomials, the resistance properties of the whole field of normally shaped sectional-area curves can be derived. This may lead to the use of further parameters as enumerated in the present report. Besides, important results for such peculiarities as bulbs and cruiser sterns already have been obtained.

8. The influence on wave resistance of the vertical distribution of the displacement (within a dimensionless shape) can be investigated with less effort. It leads to the simple rule that the displacement should be arranged as deep as possible. Thus, U-shaped sections, from the viewpoint of wave resistance, are as a rule superior to V-shaped sections; possible exceptions are mentioned.

9. The experimental check of theoretical results at low speed-length ratios is seriously hampered by the principle on which Froude's method is based as well as by experimental inconsistencies. No reasonable experimental analysis of the wave resistance of full slow hulls can be made without an additional research on viscous-form drag, especially viscous-pressure drag. However, keeping in mind these restrictions, the following conclusions can be drawn from comparisons between calculated wave resistance and measured residual resistance values 10 to 16.

10. The trends of the calculated and measured resistance curves generally agree well, but the interference effects (humps and especially hollows) are exaggerated by the theory.

11. The absolute values of the curves agree reasonably within certain ranges of L/B and ϕ ; for low L/B ratios and high prismatic coefficients at small Froude numbers the discrepancies are large. Michell's theory clearly overestimates the effects of increasing beam; unfortunately experimental data dealing with the dependence of resistance upon beam and draft are astonishingly scarce, so that neither theory nor experiment yields accurate data on resistance effects due to the most elementary changes of a hull.

12. The relative merits of different forms as established by theory are in many cases supported by experiments. However, theoretical deductions are liable to lead to exaggerations and even to errors when they are based on interference effects which can be affected by viscosity. Theory may fail completely when phenomena are discussed which depend essentially upon viscosity, such as the relative efficiencies of fore and afterbodies, unless some additional corrections are introduced.

13. The close coincidence between theoretical and experimental deductions on the effect of small changes in shape is stressed as a fact of primary importance.

14. Although experiments generally corroborate theoretical results as to the influence of the vertical distribution of displacement, there are exceptions where V-sections in the forebody are superior to U-sections at high Froude numbers contrary to calculated results. The actual optimum t values

are much lower than following from theory or Taylor's experiments. Thus the universal validity of Taylor's $\frac{R}{\Delta}$, $t, \frac{v}{\sqrt{L}}$ charts (Reference 42) must be questioned.

15. The omission of the actual model attitude in the theory causes errors at high Froude numbers.

16. The application of Michell's theory to ships with a flat bottom, although contrary to the conditions for which the integral is valid, still yields useful results. There are theoretical and experimental indications that the concept of dimensionless shape may be overstrained when it is applied to larger values of B/L or B/H .

17. Shallow-water effects can be investigated by a formula due to Sreten-sky. The basic parameters are a Froude number $F_h = v/\sqrt{gh}$ and the ratio of depth of water to model length h/L ; the ratio H/h is less characteristic as long as it is not close to unity.

18. A further integral valid for the wave resistance of ships moving in a rectangular canal yields information on the permissible model sizes for different towing basins. Earlier data based only on the ratio of the cross section of the models to the cross section of the basins are generally insufficient. Correction factors can be derived for converting model results to full size; besides F_h and h/L , the ratios of model length to basin width, L/b , and of basin depth to basin width are characteristic parameters.

19. The simultaneous treatment of the ship and the propeller leads to important results on the interaction between the hull and the propeller, including wave phenomena.

20. The resistance of wholly submerged bodies, especially bodies of revolution, when running close to the surface can be treated on similar lines to those of ships.

21. A short synopsis of methods is given which have been proposed by different authors with the intention of improving Michell's theory of wave resistance. The direction of these aims is given by the serious restrictions of Michell's theory:

- a. Assumption of a frictionless liquid.
- b. Assumption of a wedgelike form.

c. Assumption of a fixed position (neglect of bodily rise or sink and trim).

d. Assumption of a small height-length ratio of waves created by the ship including neglect of the ship form above the water when computing pressure.

ACKNOWLEDGMENT

This report presents only a skeleton of the subject; by studying the numerous references which, however, do not include important experimental work, it is possible to fill up to some extent the rather sketchy picture. Further, it is hoped that some of the projects now under way at the David Taylor Model Basin will lead to broader applications of the theory. This part of the research work, due to its magnitude, must be carried out by large institutions; the development so far has been hampered by the fact that individuals interested in the problems have only occasionally found the necessary support.

The present investigation has been initiated and sponsored by the Director of the Taylor Model Basin, Rear Admiral C.O. Kell, USN, the former head of the Hydromechanics Laboratory, Captain F.X. Forest, USN, and the Chief Naval Architect, Dr. F.H. Todd, to whom the author wishes to express his sincere thanks. Dr. Todd's help was essential in completing this ample report.

APPENDIX 1

THE EQUATION OF THE SHIP SURFACE

The axes are chosen as shown in Figure 1. Then the equation of the hull surface may be written

$$y = \pm y(x, z) \quad [1]$$

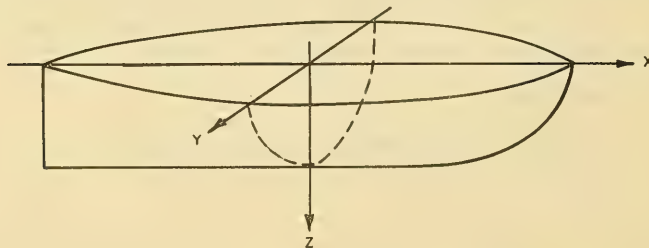


Figure 1 - Axes of Reference

The double sign appears because the hull consists of two essentially symmetrical halves. In most cases it is sufficient to consider

$$y = +y(x, z) \quad [1a]$$

and to double the results.

When dealing with resistance problems it is advantageous to let the X, Y plane coincide with the free surface and the Y, Z plane with the midship section. Calculations of buoyancy, however, are commonly performed starting from the keel. Some differences in definitions and denotations arise because of the two systems mentioned, but in our present calculations we use only the coordinates in accordance with Figure 1. The basic elements of the form can be expressed as follows:

$$1. \text{ Load water line } y(x, 0) = X[x] \quad [2]$$

$$2. \text{ Midship section } y(0, Z) = Z[z] \quad [3]$$

$$3. \text{ Longitudinal section } y(x, z) = 0 \quad z = k[x] \quad [4]$$

$$4. \text{ Sectional-area curve } A[x] = 2 \int_0^{k[x]} y(x, z) dz \quad [5]$$

for a rectangular center-plane contour

$$A[x] = 2 \int_0^H y(x, z) dz \quad [6]$$

Brackets [] are used to distinguish operations performed on dimensional coordinates. The volume is given by

$$\nabla = 2 \int_{-l}^{+l} \int_0^{k[x]} y dx \, dz = \int_{-l}^{+l} A[x] dx \quad [7]$$

Dimensionless coordinates are introduced:

$$\xi = x/l \quad \eta = \frac{y}{b} \quad \zeta = \frac{z}{H} \quad [8]$$

Thus

$$y = y(x, z) = b\eta(\xi, \zeta) \quad [9]$$

and the surface is described by its nondimensional shape,

$$\eta(\xi, \zeta)$$

a principal dimension and the ratio B/L and H/L . The nondimensional basic curves are defined as follows:

$$1. \text{ Load water line } \eta(\xi, 0) = X(\xi) = \eta_0(\xi) \quad [10]$$

$$2. \text{ Midship section } \eta(0, \zeta) = Z(\zeta) = \eta_0(\zeta) \quad [11]$$

$$3. \text{ Longitudinal section } \eta(\xi, \zeta) = 0 \quad \zeta = K(\xi) \quad [12]$$

The area coefficients and coefficients of fineness are

$$\alpha = \frac{1}{2}(\alpha_A + \alpha_F) = \frac{1}{2} \left[\int_{-1}^0 X(\xi) d\xi + \int_0^1 X(\xi) d\xi \right] \quad [13]$$

$$\beta = \int_0^1 Z(\zeta) d\zeta \quad [14]$$

$$\delta = \frac{1}{2}(\delta_A + \delta_F) = \frac{1}{2} \left[\int_{-1}^0 \int \eta(\xi, \zeta) d\xi d\zeta + \int_0^1 \int \eta(\xi, \zeta) d\xi d\zeta \right] \quad [15]$$

The areas of sections are given by

$$A(\xi) = 2 \int_0^{K(\xi)} \eta(\xi, \zeta) d\zeta$$

However, we define a slightly different dimensionless expression as sectional-area curve

$$A^+(\xi) = \frac{1}{2\beta} A(\xi) = \frac{1}{\beta} \int_0^{K(\xi)} \eta(\xi, \zeta) d\zeta \quad [16]$$

Throughout the text the symbol η will be used for equations of surfaces as well as lines. In exceptional cases we add the symbols of independent variables to avoid ambiguity.

Ship lines and surfaces can be split up, with respect to the midship section, into a main symmetrical (even) part $\eta_s(\xi)$ and a secondary asymmetrical (odd) part $\eta_a(\xi)$. Only η_s contributes to the total area or volume of the curve or surface.

$$\eta = \eta_s + \eta_a \quad [17]$$

$$\int_{-1}^{+1} \eta d\xi = 2 \int_0^1 \eta_s d\xi \quad [18]$$

The asymmetrical part alone contributes to the static moment with respect to the midship section

$$S_\eta = \int_{-1}^{+1} \xi \eta d\xi = 2 \int_0^1 \xi \eta_a d\xi \quad [19]$$

The position of a centroid $x_0/l = \xi_0$ is given by

$$\xi_0 = \frac{S_\eta}{2\alpha} = \frac{\int_0^1 \xi \eta_a d\xi}{\int_0^1 \xi \eta_* d\xi} \quad [20]$$

We define an "elementary ship" as a hull shape described by (1), the equation

$$\eta(\xi, \zeta) = X(\xi)Z(\zeta) \quad [21]$$

and (2) a rectangular center-plane contour. Dimensionless surface ordinates are obtained as products of the corresponding LWL and midship-section ordinates. Elementary ships are characterized by the following properties mentioned earlier:

$$(a) \quad A^+(\xi) = X(\xi) \quad [22]$$

i.e., the dimensionless water line and sectional-area curves are identical; hence

$$(b) \quad \phi = \alpha \quad \delta = \alpha\beta \quad [23]$$

(c) All sections are affine to the midship section. It is an advantage of dimensionless representation that integral properties like area coefficients of curves, etc., are invariant to affine transformations.

In principle any continuous ship surface can be expressed by a polynomial. This follows from Weierstrass' theorem: A continuous function $y(x, z)$ within prescribed boundaries can be approximated with any desired degree of accuracy by a polynomial in x, z . Thus

$$\begin{aligned} y &= \sum A_{mn} x^n z^m \\ \eta &= \sum a_{mn} \xi^n \zeta^m \end{aligned} \quad [24]$$

are general expressions for the ship surface. Instead of [24] we choose the special form

$$\eta = 1 - \sum a_{mn} \xi^n \zeta^m \quad [25]^*$$

*The minus sign before Σ has been introduced by analogy with Chapman's parabolas. From a mathematical point of view the plus sign is preferable in all formulas like [25], [27], [28], etc.

When the center plane contour is a rectangle even equations with a small number of terms yield satisfactory results. When dealing with more general cases, it may be advantageous to introduce fractional powers. However, we need not consider such procedures within the scope of the present report.

Derivatives of dimensional and dimensionless values are connected by

$$\frac{dy}{dx} = \frac{B}{L} \frac{d\eta}{d\xi} \quad [26]$$

water lines and sectional-area curves are expressed by

$$X(\xi) = \eta(\xi) = 1 - \sum a_n \xi^n \quad [27]$$

sections by

$$Z(\xi) = \eta(\xi) = 1 - \sum b_m \xi^m \quad [28]$$

where generally it is assumed that

$$\eta(\xi) = 0 \text{ for } \xi = \pm 1$$

$$\sum a_n = 1 \quad [29]$$

Families with two arbitrary parameters are named basic systems. These parameters can be expressed by ϕ or α and Taylor's t -coefficient where

$$t = - \left| \frac{\partial \eta}{\partial \xi} \right|_{\xi = 1} \quad [30]$$

We use the symbol

$$(n_1, n_2, n_3; \phi; t) = 1 - \sum_{n_1 n_2 n_3} a_{n_i} \xi^{n_i} \quad [31]$$

with the condition

$$\sum a_{n_1} = 1$$

α may be substituted for ϕ .

Further form parameters are

$$K = \left| \frac{\partial^2 \eta}{\partial \xi^2} \right|_{\xi = 0} \quad [32]$$

- (3) the dimensionless curvature at the midship section of the ship line,
- (4) Length of parallel middlebody,
- (5) Position of the centroid of the fore or aft area (volume), for instance

$$\frac{\int_0^1 \eta \xi d\xi}{\alpha_F} \quad [33]$$

(6) The corresponding moment of inertia coefficients

$$\int_0^1 \eta \xi^2 d\xi \quad [34]$$

(7) Higher degree moments

$$\int \eta \xi^3 d\xi \quad [35]$$

(8) Position of the point of inflection of the curve. Only conditions (3) and (4) have been used in the present investigation; the latter has been considered indirectly by varying the degree of the basic form and the former by using special polynomials.

Various examples of basic families are reproduced in Figures 15 to 17, in addition some curves of $\frac{\partial \eta}{\partial \xi}$ and $\frac{\partial^2 \eta}{\partial \xi^2}$ are shown in Figure 38. Introducing explicitly the parameters ϕ and t , the equation of a family can be put into the form

$$\eta = f_0(\xi) + \phi f_1(\xi) + t f_2(\xi)$$

or

$$(n_1 n_2 n_3; 0; 0) + \phi (n_1 n_2 n_3; 1; 0) + t (n_1 n_2 n_3; 0; 1) \quad [36]$$

$f_1(\xi) = (n_1 n_2 n_3; 1; 0)$ complies with the conditions

$$\begin{aligned} - \left| \frac{\partial f_1(\xi)}{\partial \xi} \right|_{\xi=1} &= t_1 = 0 \\ \int_0^1 f_1(\xi) d\xi &= 1 \end{aligned} \quad [36a]$$

$f_2(\xi) = (n_1 n_2 n_3; 0; 1)$ with the conditions

$$\begin{aligned} - \left| \frac{\partial f_2(\xi)}{\partial \xi} \right|_{\xi=1} &= t_2 = 1 \\ \int_0^1 f_2(\xi) d\xi &= 0 \end{aligned} \quad [36b]$$

Formulas of the three functions involved are given for various values of $n_1 n_2 n_3$ and the graphs drawn in Figure 18; for convenience the function $(n_1 n_2 n_3; 0.1; 0) = 0.1(n_1 n_2 n_3; 1; 0)$ has been introduced.

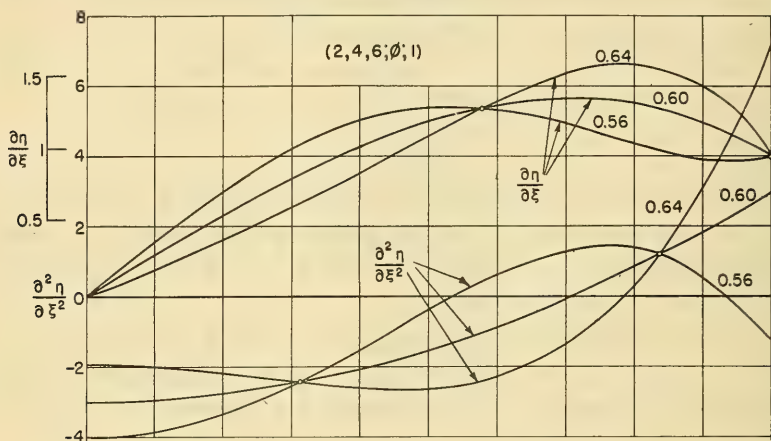


Figure 38a

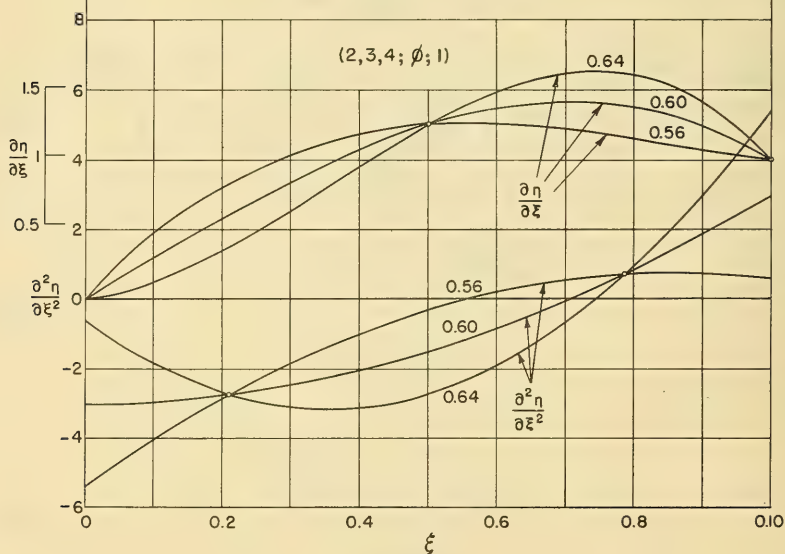


Figure 38b

Figure 38 - $\frac{\partial \eta}{\partial \xi}$ and $\frac{\partial^2 \eta}{\partial \xi^2}$ for Some Ship Lines

Finer variations of form such as changes of the curvature may be performed by polynomials of the type $(n_1 n_2 n_3 n_4; 0; 0)$ for instance

$$(2, 3, 4, 6; 0; 0) = \xi^2 - \frac{32}{9}\xi^3 + \frac{10}{3}\xi^4 - \frac{7}{9}\xi^6 \quad [37]$$

or

$$(2, 4, 6, 8; 0; 0) = \xi^2 - 5\xi^4 + 7\xi^6 - 3\xi^8 \quad [38]$$

complying with the conditions

$$\int_0^1 (\quad) d\xi = 0; \left| \frac{\partial}{\partial \xi} (\quad) \right|_{\xi=1} = 0$$

and

$$\left| (\quad) \right|_{\substack{\xi=0 \\ \xi=1}} = 0 \quad [39]$$

(Cf. Reference 102). The presence of the square ξ^2 is obviously essential when investigating the curvature of the midship section.

Additional even functions are introduced by using odd powers of absolute values. Especially the cube $|\xi|^3$ is important for various purposes. We normally omit the symbol $| \quad |$.

When different equations are used for the forebody η_F and afterbody η_A the symmetrical part of the resulting body is given by $1/2(\eta_F + \eta_A)$, the asymmetrical by

$$\pm \frac{1}{2}(\eta_F - \eta_A) \quad [40]$$

The system of curves representing water lines and sectional area curves differs from Taylor's system⁴⁴ by:

A. The choice of axes of reference. The origin is located at the midship section; the fore and afterbodies can be dealt with simultaneously even if expressed by different equations.

B. A greater variety of forms.

C. The final purpose of obtaining explicit equations of the surface, not of sets of lines.

D. Parallel middlebodies may be introduced. However, when using equations of surfaces for resistance calculations it is generally simpler to approximate a cylindrical part by high powers of ξ .

Surfaces of a more general character than these elementary ships can be deduced in various ways. For rectangular contours surface equations can be derived immediately from the expression [25]. However, it seems preferable to begin with simple geometrical concepts.

A. Consider for instance

$$\eta = [X(\xi) - v(\xi)f_3(\xi)] Z(\xi) \quad [41]$$

where the "fining function" $v(\xi)$ complies with the condition

$$v(1) = v(-1) = v(0) = 0$$

$$f_3(0) = 0$$

[41] can be interpreted as an elementary ship minus a layer $v(\xi)f_3(\xi)Z(\xi)$ which assumes zero values on the center-line contour. The equation of the sectional-area curve becomes

$$A^+(\xi) = X(\xi) - \frac{\beta_2}{\beta} v(\xi) \quad [42]$$

with

$$\beta_2 = \int_0^1 Z(\xi)f_3(\xi)d\xi$$

Putting for instance $f(\xi) = \xi$ we get an inclination of sections at the LWL. When $v(\xi) \geq 0$, $\beta_2 > 0$, Equation [42] expresses the fact that the sectional-area curve becomes finer than the water line towards the ends. The local sectional-area coefficients are expressed by

$$\beta(\xi) = \beta - \beta_2 \frac{v(\xi)}{X(\xi)} \quad [43]$$

i.e., they are smaller than β when $\beta_2 > 0$, $v(\xi) > 0$.

B. An additional effect is obtained by substituting for $Z(\xi)$

$$Z(\xi, \xi) = 1 - b_{n_1}(\xi)\xi^{n_1} \dots \quad [44]$$

By suitably choosing the functions $b_{n_1}(\xi)$ and exponents n_1 the fullness and maximum curvature of sections towards the ends can be reduced.

When a parallel middlebody is inserted in the forebody of length ξ_F an appropriate expression for a water line is

$$\eta = 1 - \sum a_n \left(\frac{\xi - \xi_F}{1 - \xi_F} \right)^n \quad [45]$$

Similarly for the afterbody with a parallel length ξ_A

$$\eta = 1 - \sum a_n \left(\frac{\xi - |\xi_A|}{1 - |\xi_A|} \right)^n \quad [46]$$

APPENDIX 2

FORMULAS FOR THE WAVE RESISTANCE

All formulas here collected are valid for uniform motion only.

A. The Wave Resistance of a Source or Sink*

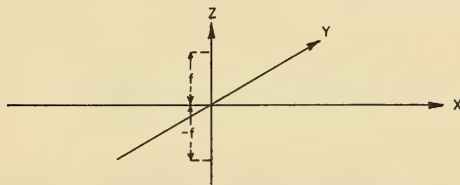


Figure 39 - Source (sink) Beneath the Surface

The flow due to a single source or sink has been studied by Havelock.⁶¹ Assume the usual axes but with OZ directed vertically upwards (that is the direction used in almost all Havelock's papers contrary to Michell's assumption).

The velocity potential of a source m moving with a constant speed v in the direction of x at a distance f beneath the surface can be written in the form

$$\phi = \frac{m}{r_1} - \frac{m}{r_2} + \phi_w \quad [1]$$

*Unfortunately there is no easily readable introduction into the theory of wave resistance in any Western language. Lamb's book stops just when matters become interesting for practice.

where

$$\begin{aligned} r_1^2 &= x^2 + y^2 + (z + f)^2 \\ r_2^2 &= x^2 + y^2 + (z - f)^2 \end{aligned} \quad [2]$$

m/r_1 represents the velocity potential of the source in an unbounded fluid,
 $-m/r_2$ is the potential of a sink of equal strength at the image point under the same conditions, and
 ϕ_w is the potential due to wave motion.

For extreme Froude numbers ϕ_w vanishes; thus the potential is given by

$$\phi = \frac{m}{r_1} - \frac{m}{r_2} \quad [3]$$

In the other limiting case, $F \rightarrow 0$, the surface acts as a rigid cover; the appropriate expression for ϕ is then

$$\phi = \frac{m}{r_1} + \frac{m}{r_2} \quad [4]$$

a term, say ϕ_w' due to waves, disappears again. The expression

$$\phi = \frac{m}{r_1} + \frac{m}{r_2} + \phi_w' \quad [5]$$

is legitimate as (1); it has been used by Dickmann.

Using the surface condition

$$\frac{\partial^2 \phi}{\partial^2 x^2} + \kappa_0 \frac{\partial \phi}{\partial z} - \mu \frac{\partial \phi}{\partial x} = 0 \quad [6]$$

(where μ is Rayleigh's friction, Reference 25, $\kappa_0 = \frac{g}{c^2}$)

and Laplace's equation

$$\Delta \phi = 0 \quad [7]$$

Havelock gives the expression for the velocity potential

$$\phi = \frac{m}{r_1} - \frac{m}{r_2} - \frac{\kappa_0 m}{\pi} \operatorname{Re} \int_{-\pi}^{+\pi} \sec^2 \theta d\theta \int_0^\infty \frac{\exp [-\kappa(f-z) + i\kappa\omega] d\kappa}{\kappa - \kappa_0 \sec^2 \theta + i\mu \sec \theta} \quad [8]$$

where Re stands for the real part. Dickmann's form yields

$$\phi = \frac{m}{r_1} + \frac{m}{r_2} - \frac{m}{\pi} \operatorname{Re} \int_{-\pi}^{+\pi} d\theta \int_0^{\infty} \frac{(\kappa + i\mu \sec \theta) \exp[-\kappa(f-z) + i\kappa\omega]}{\kappa - \kappa_0 \sec^2 \theta + i\mu \sec \theta} d\kappa \quad [9]$$

and leads to a resistance

$$R = \frac{16\pi\rho m^2}{r^2} \gamma_0 e^{-2\gamma_0} [K_0(2\gamma_0) + K_1(2\gamma_0)] \quad \gamma_0 = \frac{1}{2F^2} \quad [10]$$

where K_0, K_1 are Bessel functions of the third kind.

Tables of wave ordinates in the X,Z plane due to the motion of a source, have been calculated by Wigley.⁸⁴

B. Michell's Integral

Four different methods have been used for calculating the wave resistance of slender ship-like bodies:

1. Michell's original deduction based on the concept of a wedge-like ship depends upon the computation of pressure changes δp due to the wave pattern.

$$R = -2 \iint \delta p dy dz = -2 \iint \delta p \frac{\partial y}{\partial x} dx dz = 2\rho v \iint \frac{\partial \phi}{\partial x} \frac{\partial y}{\partial x} dx dz \quad [11]$$

or

$$R = \frac{4\rho g^2}{\pi v^2} \int_1^{\infty} (I^2 + J^2) \frac{\lambda^2}{\sqrt{\lambda^2 - 1}} d\lambda \quad [12]$$

where

$$I = \int_{-e}^{+e} \int_0^{k[x]} \frac{\partial y}{\partial x} e^{-\lambda^2 g z / v^2} \cos\left(\frac{\lambda g x}{v^2}\right) dx dz$$

$$J = \int_{-e}^{+e} \int_0^{k[x]} \frac{\partial y}{\partial x} e^{-\lambda^2 g z / v^2} \sin\left(\frac{\lambda g x}{v^2}\right) dx dz \quad [13]$$

$z = \kappa[x]$ is the equation of the longitudinal contour.

Only odd terms of the hull equation contribute to the value of I. For a symmetrical hull $I = 0$.

Introducing dimensionless coordinates $\xi = \frac{x}{l}$ and putting

$$\gamma = \gamma_0 \lambda$$

where $\gamma_0 = \frac{gL}{2v^2} = \frac{1}{2F^2}$ we obtain for a rectangular contour

$$I = HB I^+(\gamma) = HB \int_0^{+1} \int_0^{+1} \frac{\partial \eta_a}{\partial \xi} e^{-2 \frac{H}{L} \frac{\gamma^2}{\gamma_0} \xi} \cos \gamma \xi d\xi d\zeta \quad [14]$$

$$J = HB J^+(\gamma) = HB \int_0^{+1} \int_0^{+1} \frac{\partial \eta_s}{\partial \xi} e^{-2 \frac{H}{L} \frac{\gamma^2}{\gamma_0} \xi} \sin \gamma \xi d\xi d\zeta \quad [15]$$

η_s, η_a denote the symmetrical and asymmetrical parts of the hull with respect to the midship section. The resistance R can be written

$$R = \frac{8\rho g}{\pi} \frac{B^2 H^2}{L} \int_0^\infty [I^{+2}(\gamma) + J^{+2}(\gamma)] \frac{\left(\frac{\gamma}{\gamma_0}\right)^2}{\sqrt{\left(\frac{\gamma}{\gamma_0}\right)^2 - 1}} d\gamma \quad [16]$$

This form has some slight advantage compared with the original one from the point of view of computation. For rectangular contours integrations with respect to ξ and ζ can be handled independently; the discussion is based on quadratures of the type:

$$\begin{aligned} M_n(\gamma) &= \int_0^1 \xi^n \sin \xi d\xi \\ M_n^I(\gamma) &= \int_0^1 \xi^n \cos \xi d\xi \\ E_m(\gamma) &= \int_0^1 e^{-\gamma \xi} \xi^m d\xi \quad \gamma = 2 \frac{H}{L} \frac{\gamma^2}{\gamma_0} \end{aligned} \quad [17]^*$$

Especially simple expressions are obtained for the elementary ship: The integral $J^+(\gamma)$ can be written in the form

$$J^+(\gamma) = S(\gamma) \Phi(\gamma) \quad [18]$$

where $S(\gamma)$ depends only on the longitudinal and $\Phi(\gamma)$ only on the vertical distributions of the displacement. Examples are discussed in the main text.

A rigorous proof of Michell's integral based on potential theory is due to Sretensky.³⁸

Three further methods of calculating wave resistance are mainly due to Havelock. Explicit results so far have only been obtained in those cases where a ship form can be represented by images.

Using an approximate connection between the normal velocity and a plane surface distribution

$$2\pi\sigma = v \frac{\partial y}{\partial x} \quad [19]$$

*In the paper by the author JSTG (1950) there is an obvious misprint on page 418 as the double value of the integral is defined as $M_n(\gamma)$.

Havelock has converted Michell's integral into the form^{62, 72}

$$R = 16\pi \kappa_0^2 \rho \int_0^{\pi/2} (P^2 + Q^2) \sec^3 \theta d\theta \quad [20]$$

$\kappa_0 = \frac{g}{v^2}$ where

$$P = \iint \sigma \exp[i\kappa_0 x \sec \theta - \kappa_0 z \sec^2 \theta] dx dz \quad [21]$$

substituting

$$\sigma = \frac{v}{2\pi} \frac{\partial y}{\partial x}$$

Havelock's form of Michell's integral^{62, 72} is obtained

$$R = \frac{4\rho g^2}{\pi v^2} \int_0^{\pi/2} (P^2 + Q^2) \sec^3 \theta d\theta \quad [22]$$

$$P = \iint \frac{\partial y}{\partial x} e^{gz/v^2 \sec^2 \theta} \cos\left(\frac{gx}{v^2} \sec \theta\right) dx dz \quad [23]$$

$$Q = \iint \frac{\partial y}{\partial x} e^{gz/v^2 \sec^2 \theta} \sin\left(\frac{gx}{v^2} \sec \theta\right) dx dz \quad [24]$$

Compared with Michell's formula the direction of the z-axis is reversed (upwards). Havelock's form has advantages from the point of view of computation because of the finite limits of the integral.

Putting $\sec \theta = \cosh u$ a third form has been introduced by Havelock.⁷⁹

The same result can be deduced as follows:

2. Using Lamb's method based on Rayleigh's frictional coefficient μ (frictional force proportional to velocity). With the inclusion of the frictional term in the equations of fluid motion, energy is dissipated at a rate equal to 2 times the total kinetic energy of the liquid and this must be equal to the product Rv . This approach is efficient, but highly artificial.^{26, 69}

3. Establishing the connection between the wave profile at a great distance aft of a moving body and the wave resistance of the body. This method represents an extension of the usual theory of group velocity. The rate of work done on the fluid by the moving body (otherwise expressed, the power expended) is equal to the rate of work done across a fixed vertical control plane minus the rate of flow of total energy across this plane.⁷⁰

4. By the method of singularities (Lagally's theorem) sketched in the main text. The resistance is computed from

$$R = 4\pi\rho \iint \sigma(x,z) u(x,z) dx dz \quad [25]$$

where $\sigma(x, z)$ is the source-sink distribution over the vertical center plane and $u(x, z)$ the x component of fluid velocity at the point x, z , which must be calculated from $\partial\phi/\partial x$.⁷² Evaluating [25], Michell's integral is again obtained. However, the method of singularities appears to be far superior to any other known, when determining the forces experienced by singularities in a given flow; it enables us to calculate the lift and the transverse force as well as the resistance, and is especially powerful when systems of bodies are investigated. Havelock's and Dickmann's work furnished beautiful examples of its application.

We mention finally formulas developed by Kotchine⁵⁸ which have been successfully applied by him and by Haskind¹¹³ to the calculation of forces on floating or submerged bodies.

C. Sretensky's Formula for the Wave Resistance in Shallow Water

Sretensky has developed a formula for the wave resistance of "Michell's ship" in shallow water. The theory is valid under the same assumptions as Michell's integral provided the draft/depth ratio is small. The deduction is based on methods of potential theory.

$$R = \frac{8\pi\rho g}{v^2} \int_{m_0}^{\infty} \frac{P^2 + Q^2}{\sqrt{m^2 - \frac{gm}{v^2}} \tanh mh} \frac{m \, dm}{\cosh mh} \quad [26]$$

where

$$P = \iint \cosh m(z + h) \cos \sqrt{\frac{gm}{v^2}} \tanh mh \times \sigma(x, z) dx dz \quad [27]$$

$$Q = \iint \cosh m(z + h) \sin \sqrt{\frac{gm}{v^2}} \tanh mh \times \sigma(x, z) dx dz \quad [28]$$

$\sigma(x, z)$ is the source-sink distribution which can generate a ship. Using the relation between the normal velocity component and source strength:

$$2\pi\sigma = v \frac{\partial y}{\partial x}$$

the integral can be easily rewritten in terms of the surface equation $y(x, z)$.

The lower limit of the integral m_0 is of special interest; it is given by the equation

$$\tanh m_0 h = \frac{v^2}{gh} m_0 h \quad [29]$$

$$\tanh m_0 h = F_h^2 m_0 h \quad [29a]$$

The roots of [29a] can be represented as functions of the parameter

F_h .

When the critical velocity $F_h^2 = 1$ is reached $m_0 h$ becomes zero and remains zero for $F_h^2 > 1$. Thus for the whole supercritical range $m_0 = 0$. The integral (2b) can be put into various shapes when performing actual computations.

D. The Wave Resistance of Ships in Rectangular Canals

The rather complicated formula for this case has been derived by Sedov and Keldysh and Sretensky.^{38,39}

$$R = \frac{16\pi^2 \rho g}{b v^2} \left[P_0^2 + Q_0^2 + 2 \sum_1^{\infty} (P_k^2 + Q_k^2) \right]$$

where

$$Q_k = \frac{\iint \cosh m_k(z+h) \sin \sqrt{\frac{gm_k}{v^2} \tanh m_k h} x \sigma(x,z) dx dz}{\sqrt{\left(1 + \frac{4\pi^2 K^2}{b^2 m_k^2}\right)} \cosh^2 m_k h - \frac{gh}{v^2}}$$

P_k is a similar expression with the cosine instead of the sin and represents the influence of odd terms of the surface equation, k is an integer while b is the width of the canal.

The values of m_k are the roots of the equations

$$\frac{b}{2} \sqrt{m^2 - \frac{gm}{v^2} \tanh mh} = \pi K$$

APPENDIX 3

THE VISCOUS PRESSURE RESISTANCE

Observations and some measurements of the velocity distribution at the stern indicate that with normal models the chief reason for the viscous pressure resistance is, contrary to earlier opinions, not the separation but the pressure defect due to increase of boundary layer thickness.

Two methods have been proposed to deal with this problem in a quantitative way:



Figure 40 - Scheme for Computing Viscous Pressure Drag

1. It is assumed that the frictional layer does not disturb seriously the potential flow outside of it. The static pressure $p(A')$ at the boundary point A' of the frictional layer is calculated and, using an essential property of this layer, the assumption is made that the pressure on the body (Point A) is equal to $p(A')$. It is claimed that the viscous resistance so computed agrees well with experiments.⁷

2. It is assumed that the body and the frictional wake form a new effective body, for which the pressure distribution may be calculated from considerations of potential flow. This procedure leads again to a pressure drop at the stern compared with the calculation applied for the original body.³⁰

Obviously these two methods can be only applied to infinitely long cylinders (two-dimensional case) and to bodies of revolution, for which the boundary layer thickness can be calculated. Thus it seems to be natural that any research on viscous pressure resistance starts with investigations on these bodies; a body of revolution is obviously a closer approximation to a double model than an infinitely long cylinder.

As has been pointed out, experimental data are very scarce. The following empirical formulas are quoted:

1. Two-dimensional case (symmetrical profiles)

Hoerner's formula for the total viscous drag²²

$$\frac{C_R}{C_f} = 1 + 2 \frac{d}{L} + 70 \left(\frac{d}{L} \right)^4 \quad \alpha\text{-thickness} \quad [1]$$

2. Axial symmetry (bodies of revolution)

Hoerner's formula for the total viscous drag²²

$$\frac{C_R}{C_f} = 1 + 0.5 \frac{d}{L} + b \left(\frac{d}{L} \right)^4 \quad d\text{-diameter} \quad [2]$$

For double models and bodies of revolution a formula has been given by Weinig.⁵⁰

It is obviously erroneous to calculate the ratio of viscous pressure resistance of ship forms to corresponding total viscous drag results valid for cylinders of the same B/L ratios as can be found even in serious books.

The concept "fairness of lines" which includes some postulates as to the continuity of their derivatives has been developed only empirically.

The present theories of wave resistance do not yield any answer as to the order of "smoothness of lines" required since they deal really with image distributions, not with the actual ship form.

There exist, however, some results on the influence of discontinuity in curvature on the pressure distribution in the two-dimensional case. At such points the pressure curve is characterized by a vertical tangent, i.e., a sudden change in pressure must be expected. When these critical points lie in a region of rising pressure, an increasing tendency to separation may be expected.¹¹⁴

Even a discontinuity in the third derivative of a ship line leads to an indentation in the pressure curve; it is, however, less probable that the boundary layer may be adversely affected by it.¹¹⁴

Our actual knowledge as to how the ship resistance depends on such peculiarity of lines is practically nil.

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LIST OF ABBREVIATIONS

TINA	Transactions of the Institution of Naval Architects (London)
TSNAME	Transactions of the Society of Naval Architects and Marine Engineers (New York)
J.S.T.G.	Jahrbuch der Schiffbautechnischen Gesellschaft (Berlin)
TNECIMS	Transactions of the North-East Coast Institution of Marine Engineers and Shipbuilders (Newcastle)
Proc. Royal Soc.	Proceedings of the Royal Society, A, (London)
ZAMM	Zeitschrift für Angewandte Mathematik und Mechanik (Berlin)

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